

Comparison of different fast multipole methods for solving the radiosity equation

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Abstract. For the radiosity equation, we investigate iterative solutions and acceleration by the Fast Multipole Method (FMM). A new fast method was introduced recently by the authors [ENU-MATH 2005] based on the method proposed by Gimbutas and Rokhlin [SIAM J. Sci. Comput., 24, 796–817 (2002)]. We present here an optimized algorithm of this fast method for a kernel invariant by translation which satisfies some additional symmetries for $1/r^7$. Finally, theoretical and numerical comparison of different FMM methods is presented for the radiosity kernel based on $1/r^4$ for a complete multilevel fast multipole method.

1. INTRODUCTION. Fast methods for solving the radiosity equation is considered by using the Fast Multipole Method (FMM), in the context of heat transfer calculations. This equation, which is an integral equation, models radiative exchanges between gray diffuse surfaces without participating media [10]. The radiosity equation plays also an important role in obtaining realistic image in computer graphics [17]. After discretization of the whole surface by finite elements, the size of the system generated can be quite large, and consequently the cost of solving this system is time consuming (with an iterative method: $O(N^2)$ where N is the number of elements) and memory. Three classes of fast methods to solve this problem have been developed. Firstly, classical hierarchical methods (HM) for sets of plane surfaces and their generalization to initial curved surfaces with clustering [17] in computer graphics, and secondly a \mathcal{H} -matrix with an adaptative cross approximation (ACA) in heat transfer [2] and thirdly methods based on an expansion of the integral kernel: panel clustering [1] and FMM [8, 11] in computer graphics. The main drawback of HM and panel clustering is their limitation to sets of plane surfaces. Furthermore, the first class encounters problems of iterative robustness and of accuracy prediction which are unacceptable for application in radiative heat transfer calculations. The drawback of ACA is there is no guarantee for a good approximation. To accelerate iterative solution of the radiosity equation, we propose to use FMM as in [8, 11]. This method was introduced for N-body problems [13] and used in many other physical applications. Based on an expansion of the kernel of the integral equation, this method reduces the interaction generated by the kernel between elements of the mesh to interactions between multipole boxes and so accelerates matrix-vector products of iterative methods. By using the multi-level FMM (MLFMM), we can evaluate solution system with a cost of $O(N \ln(N))$. The radiosity kernel depends on the surface, due to the normal. So to construct a FMM independant of the surface for this equation, we need to investigate a FMM expansion for an other kernel that doesn't depend on the surface. So we are interested on a FMM expansion for $1/r^4$ as in [8, 11]. A FMM method based on a Taylor expansion for smooth kernels have been proposed in [19]. A multipole expansion based on the expansion of $1/r^7$ with the Gegenbauer polynomials was used to solve the Fokker-Planck-Landau's equation ($\gamma = 3$) [9] and the radiosity equation ($\gamma = 4$) [7] with Spherical Harmonics (SH). With the multipole expansion in [7] and a formula in [16], Karapurkar *et al.* introduced a FMM expansion for the radiosity kernel with SH [8]. The Rotational Coaxial Translation Decomposition (RTCD) of [6], primary used for Laplace's equation ($\gamma = 1$), uses properties of SH to accelerate transfers between boxes. The RTCD was extended in [11] to the radiosity kernel to improve the FMM proposed in [8]. In [11, 12], we have discussed a new fast method for general kernels inspired by [5]: Reduced SVD method (R-SVD). The optimizations provided compared to [5], are based firstly on the use of a reduced SVD, and secondly on the use of symetries allowed by this reduced SVD, which cannot be used with the original method. Previously in [11], we have introduced a comparison process for surface interaction problems that takes into account the empty boxes in the octree for different

Table 1: Number of iterations and CPU time for a relative residual of 10^{-6} .

	Southwell	GS	Hybrid GS	GMRES
iter	148703	22	9	8
time (second)	9.48	8.16	3.86	3.72

MLFMM. For $1/r^4$, we used this comparison process to compare relative errors for the transfer of the FMMs for a given box. In the present paper, we extend this comparison for the complete MLFMM error for $1/r^4$ and for the radiosity kernel.

In Section 2, we present the radiosity equation and its numerical solution (iterative solution, fast methods). In Section 3, we present FMM expansions for $1/r^\gamma$ and for radiosity equation and in Section 4, an optimized reduced SVD method for $1/r^\gamma$. Finally, in Section 5, we compare theoretically and numerically the FMMs investigated for a MLFMM error, before concluding.

2. RADIOSITY EQUATION AND NUMERICAL SOLUTION. The radiosity equation is a mathematical model for the radiative exchanges between gray diffuse surfaces without participating media. In computer graphics, it models the light transport in the same condition. This equation writes, for a polyhedral surface S of \mathbb{R}^3 :

$$\forall x \in S, \quad B(x) = \epsilon(x)\sigma T^4(x) + \rho(x) \int_S V(x, y) G(x, y) B(y) d\sigma_y, \quad (1)$$

with B the radiosity, ϵ the emissivity, ρ the reflectivity, T the temperature and σ the Stefan-Boltzmann constant. The visibility $V(x, y)$ equals 1 if the two points x and y see each other ($[x, y] \cap S = \{x, y\}$) and cancels otherwise. The radiosity kernel G is given by: $G(x, y) = \frac{(x-y) \cdot n_x (y-x) \cdot n_y}{|x-y|^4}$, where n_x is the inner unit normal to the surface S at point x . As $\int_S G(x, y) d\sigma_y \leq 1$ (equal 1 for an enclosed surface), a condition to the existence of a unique solution to (1) in $L^2(S)$ is $|\rho|_{L^\infty(S)} < 1$, which is a physical condition [1].

Finite elements discretization. After discretization of the surface S by a piecewise plane triangular surface ($S_h = \bigcup_{i=1}^N S_i$) and an approximation of the radiosity B by a piecewise constant function, we obtain with the Galerkin method:

$$(I - M)B = E, \quad \text{with } M_{ij} = \rho_i F_{ij} \text{ and } E_i = \int_{S_i} \epsilon(x)\sigma T^4(x) d\sigma_x. \quad (2)$$

F is the shape factor matrix defined by: $S_i F_{ij} = \int_{S_i} \int_{S_j} V(x, y) G(x, y) d\sigma_x d\sigma_y$ which is obtained by a Gauss-Legendre quadrature formula (For the singular integral calculation, we can use the quadrature introduced in [4] after some change of variables.) and a particular calculus of visibility. This matrix satisfies two important laws: $\sum_j F_{ij} = 1$ (for an enclosed surface) and $S_i F_{ij} = S_j F_{ji}$.

Iterative solutions. We have compared four different iterative methods for system (2): Southwell [18], Gauss-Seidel (GS), Hybrid GS [14] and GMRES [15]. The Southwell or shooting method is extensively used in computer graphics. In each step, the surface S' of the surface mesh, which has the maximum residual, interacts with the other surface which can be seen by S' . In this method, there is no matrix vector product that could be accelerated by FMM. The hybrid GS method is based on the iterate of GS and a minimization of the residual of the matrix system in l^2 norm on the line defined by two following iterates of GS (Hybridization step). For our comparisons, we consider a rectangular box in the center of a sphere (of radius $R = 1$) with the characteristics $l_x = 0.4, l_y = 0.4, l_z = 0.02$. There are 7000 elements, $\rho = 0.7$ for both surfaces and $T = 80K$ for the sphere and $T = 80.2K$ for the box. The results in Table 1 show that GMRES and Hybrid GS are the fastest iterative solutions. For different configurations (ρ, T, S), we obtain the same conclusion. So these methods are the best to be used with the FMM.

Fast methods. If a good accuracy is needed for the radiosity problem, system (2) can become quite large. For the unoccluded case ($V \equiv 1$), the cost of solving (2) by an iterative method is $O(N^2)$, where N is the number of elements. If $V \neq 1$, this cost is $O(N^3)$ due to the calculus of F . Three classes

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of fast methods to solve this problem have been developed. Firstly, classical hierarchical methods (HM) for sets of plane surfaces and their generalization to initial curved surfaces with clustering [17] in computer graphics, and secondly a \mathcal{H} -matrix approximation, constructing with ACA, of the system matrix in heat transfer [2] and thirdly methods based on an expansion of the integral kernel: panel clustering and FMM [1, 8] in computer graphics. In the first class, a hierarchical representation of interactions between initial plane surfaces (or shape factor matrix) is constructed by adaptively subdividing planar surfaces into sub-surfaces according to a local error of interaction between two surfaces, to have a multiresolution solution of (2). The cost of this method is linear with respect to the refined surfaces, but quadratic in the initial plane surfaces. An improvement of HM, clustering has a quasi-linear cost in the initial plane surfaces by grouping elements into volume clusters. The drawback of HM and panel clustering are their limitation to sets of larger plane surface. Furthermore, HM and clustering encounter problems of iterative robustness and of accuracy prediction which are unacceptable in radiative heat transfer. In ACA, the original matrix for two admissible blocks is approximated by a combination of a few rows and a few columns of this matrix and there is no guarantee for a good approximation (see [3]).

3. FMM AND KERNEL EXPANSION. We propose to use the FMM applied to the radiosity equation as in [8]. The FMM was introduced for N-body problems [13] and used in many other physical applications. The potential field is decomposed into a far field and a near field using an octree decomposition of the 3D domain. The near field for a body target corresponds to the union of the box containing the target and of the neighbouring boxes. A classical calculus of interaction is used for this field. Based on an expansion of the potential of interaction (an integral kernel in our case), the FMM reduces the far interactions to interactions between multipole boxes. In the sequel, we present the different steps of MLFMM or FMM methods with the following abbreviations: S: source, M: multipole, L: local, and T: target. A FMM expansion for a kernel K approximates interactions between two points x and y inside boxes B_{x_0} and B_{y_0} respectively of centers x_0 and y_0 , by a separation of variables x and y : mainly $K(x, y) \simeq \sum_i L_i(x, x_0) \sum_{i'} T_{i,i'}(x_0, y_0) M_{i'}(y, y_0)$, where $T_{i,i'}$ is the transfer or M2L operator. The aim of the multilevel version of FMM (MLFMM) is to reduce the numerical complexity by reducing the number of transfer operations. These transfers are made at higher level. Therefore, we need an operator between father and child boxes of centers x_0^f and x_0 respectively for functions L_i (L2L operator) and M_i (M2M operator). For example, for function L_i , we have $L_i(x, x_0^f) \simeq \sum_{i'} L_{i,i'}(x_0^f, x_0) L_{i'}(x, x_0)$. The steps S2M and L2T correspond respectively to calculate $M_{i'}(y, y_0)$ and to convert $\sum_{i'} T_{i,i'}(x_0, y_0) M_{i'}(y, y_0)$ in the approximation of $K(x, y)$ given before. All FMM expansions presented here are also MLFMM expansions (see [19, 8]).

The radiosity kernel depends on the surface, due to the normal. So to construct a FMM independant of the surface for this equation, we need to investigate a FMM expansion for an other kernel that doesn't depend on the surface. As in [8], we decompose the radiosity kernel as: $G(x, y) = \sum_{h=1}^{16} \frac{TM_h(x)SM_h(y)}{|x-y|^4}$, and we obtain with $TM(x) = (x, -n_x(x.n_x), (x.n_x), n_x x^t)^t$ and $SM(y) = (-n_y(y.n_y), y, (y.n_y), y n_y^t)^t$:

$$\int_S G(x, y) q(y) d\sigma_y = \sum_{h=1}^{16} TM_h(x) \int_S \frac{Q_h(y)}{|x-y|^4} d\sigma_y, \text{ where } Q_h(y) = SM_h(y) q(y).$$

In this paper, we investigate FMM expansion for the kernel $1/|x-y|^4$. If the visibility $V \equiv 1$, we can evaluate solution system with a quasi-linear cost with the MLFMM. We consider in the following a more general potential $1/r^\gamma$, $\gamma \in \mathbb{N}^*$.

A Taylor FMM expansion of order L' [19] for a smooth kernel is given by:

$$\forall x \in B_{x_0}, \forall y \in B_{y_0}, K(x, y) \simeq \sum_{\alpha+\beta \leq L'-1} \frac{D_x^\alpha D_y^\beta K(x_0, y_0)}{\alpha! \beta!} (x - x_0)^\alpha (y - y_0)^\beta.$$

Alternatively, it is possible to construct a multipole method (MM) with an expansion with the Gegenbauer polynomials used by Lemou to accelerate the solution of Fokker-Planck-Landau's equation ($\gamma = 3$) [9]. We can obtain the same expansion expressed with Spherical Harmonics (SH) used by Hausner [7], using [16] for light transport. With a formula in [16], we obtain a SH based MLFMM expansion (see [12] for more details) which is used for Laplace's equation in 3D ($\gamma = 1$) [6] and for the

radiosity equation ($\gamma = 4$) by Karapurkar et al. [8]. the Rotational Coaxial Translation Decomposition (RTCD), primary used for $\gamma = 1$, uses properties of SH to speed up transfer. In [12], we have extended the RTCD for $\gamma \in \mathbb{N}^*$.

A MLFMM for general kernels was introduced in [11, 12] which is an optimization of the method in [5] by using a reduced SVD (R-SVD). The kernel is approximated by an expansion based on a tensor product of Legendre polynomials, at each level of the octree, and use a SVD of these approximations to construct multipole and local expansions.

4. REDUCED SVD METHOD. R-SVD was introduced in \mathbb{R}^d [11] and is inspired by the method proposed in [5]. The kernel K is, as in [5], approximated by an expansion of tensor product of Legendre polynomials: $\tilde{K}_l(x, y)$ for each level l of the octree (see [12] for a comparison of R-SVD and the method in [5]). In this method, we need two other expansions: a pseudo multipole expansion (PM) and a pseudo local expansion (PL). For a symmetrical kernel $K(x, y) = K(y, x)$, a SVD of these approximations defined for x in box b and for y in the interaction list of b is used to perform PM2PL operations (corresponding to the creation of PM2M, M2L and L2PL steps).

In this part, we will assume that all charges (q) are located in $D = [0, 1]^3$. We introduce for a box of the octree $Y^b = \prod_{h=1}^3 [b_h, b'_h]$ the multi-index $b = (l, I)$ where l is the level and I is the index of the box. In the paper, we will denote also b for the box; X^b the union of boxes at level l which are not b and immediate neighbours of b at level l , and X_2^b the union of all boxes at level l whose father is a neighbour of b 's father and which is not immediate neighbours of b (*interaction list*, see for more details [5] (*List 2*)). We have $X^b = \bigcup_{c \in a(b)} X_2^c$, where $a(b)$ is b and all b 's ancestors. We introduce also $P_m^{\alpha, \beta}$ the m th Legendre polynomial on the interval $[\alpha, \beta]$ (degree m). At level l , we approximate K by \tilde{K}_l which is defined on $Y^b \times Y^c$ with a tensor product of Legendre polynomials of maximum degree $n-1$ by direction:

$$\tilde{K}_l(x, y) = \sum_{j=0}^{n^3-1} \sum_{j'=0}^{n^3-1} \left(\int_{Y^b \times Y^c} K(x', y') P_j^b(x') P_{j'}^c(y') dx' dy' \right) P_j^b(x) P_{j'}^c(y),$$

where $P_j^b(x) = \prod_{h=1}^3 P_{j_h}^{b_h, b'_h}(x_h)$ with $j = j_1 + nj_2 + n^2 j_3$ ($j_h = 0, \dots, n-1$) and $x = (x_1, x_2, x_3)$. It corresponds also to an interpolation by tensor product of Lagrange's polynomials defined by the n Gauss points per direction of the two boxes [20].

A SVD is then considered to compress the resulting representation of \tilde{K}_l by a truncation with p terms of the SVD (exact for $p = n^3$). Then for b in level l , a SVD of \tilde{K}_l on $Y^b \times X_2^b$ is precalculated and stored: $\tilde{K}_l(x, y) \simeq \sum_{k=1}^p u_k^b(x) s_k^b v_k^b(y)$. The calculation of the SVD of \tilde{K}_l is developed in [20] and is equivalent to a calculation of interactions, for each box b , between n^3 and $n_b n^3$ particules, where $n_b = 189$ (refer to reduced SVD method because X_2^b is replaced by X^b [11, 12]). For a radial kernel invariant by translation and which satisfies $K(\lambda x, \lambda y) = g(\lambda) K(x, y)$ ($1/r^\gamma$ have these properties), it is possible to perform one SVD between one box and 16 boxes. In this R-SVD (16 R-SVD), the boxes in the interaction list of center $(x, y, z) \geq 0$ for a box of center $(0, 0, 0)$ are separated in 6 zones of boxes obtained by using the 5 symetries that inverted the coordinate applicated to the zone of 16 boxes defined by $z \leq y \leq x$ where (x, y, z) is the center of the box. Each zone generates a set of 8 zones with the composition of symetries of plane (0xy), (0xz) and (0yz) ($Z_i^b = \bigcup_{h=1,8} Z_{i,h}^b$). Now, we describe how to obtain with the 16 R-SVD an approximation of the far field potential at the point x in b : $\int_{X^b} K(x, y) q(y) dy$.

S2PM: we transform the sources for finest boxes ($l = l_{\max}$) in pseudo multipole expansion γ_j^b defined by $\gamma_j^b = \sum_{i=1}^{s_b} P_j^b(x_i) q_i \simeq \int_{Y^b} P_j^b(x) q(x) dx$, where s_b is the number of sources in the box b .

PM2PM: calculation of γ_j^b for all boxes at every level $l > l_{\max}$.
do $i = 1, 6$,

PM2M: conversion of γ_j^b into multipole expansion for all boxes for the eight zones (h) of the i^{th} set of zone defined by $M_k^{b,i,h} = \sum_{j=0}^{n^3-1} a_j^{k,i,h} \gamma_j^b$, where $a_j^{k,i,h} = \langle u_k^{b,i,h}, P_j^b \rangle_{L^2(Y^b)}$.

M2L: conversion of the multipole expansion into local expansion which is an approximation of $L_k^{b,i,h} = \sum_{c \in Z_{i,h}^b} \left(\sum_{k'=1}^p \langle u_k^{b,i,h}, u_{k'}^{b,i,h'} \rangle_{L^2(Y^c)} M_{k'}^{b,i,h} \right) \simeq \int_{Z_{i,h}^b} v_k^{b,i,h}(y) q(y) dy$. So we obtain this

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approximation: $\forall x \in Y^b, \int_{Z_{i,h}^b} k(x,y)q(y)dy \simeq \sum_{k=1}^p u_k^{b,i,h}(x) s_k^b L_k^{b,i,h}$.

L2PL: conversion of $(L_k^{b,i,h})_{h=1,8}$ into $\Gamma_j^{b,i}$, the pseudo local expansion due to charges in Z_i^b .

End do

PL2PL: calculation of the pseudo local expansion $\tilde{\Gamma}_j^b$ due to charges in X^b : $\int_{X^b} k(x,y)q(y)dy \simeq \sum_{c \in a(b)} \sum_{j=0}^{n^3-1} \Gamma_j^c P_j^c(x) = \sum_{j=0}^{n^3-1} \tilde{\Gamma}_j^b P_j^b(x)$, where $\Gamma_j^c = \sum_{i=1}^6 \Gamma_j^{c,i}$.

PL2T: evaluation of the far field potential with this latest formula for finest boxes for the target particule.

Numerical complexity. With B_l the number of non empty boxes at level l and C_{nf} the average number of boxes in the nearfield for the finest boxes, s the average number of charges in the finest boxes, and α the average number of boxes in the interaction list, the cost of the new method is: $C_{\text{nf}} N s + 2C' N n^3 + \left(\sum_{l=4}^{l_{\text{max}}} B_l \right) 3n^3 \frac{n+1}{2} + \left(\sum_{l=3}^{l_{\text{max}}} B_l \right) (\alpha p^2 + C_{\text{pml}})$, where $\sum_{l=3}^{l_{\text{max}}} B_l \simeq \sum_{l=4}^{l_{\text{max}}} B_l \simeq 8N/(7s)$, The different terms are respectively the cost of near interactions, S2PM+PL2T (C' includes the cost of calculus for $P_j^b(x_i)$, PM2PM+PL2PL, PM2M+L2PL ($C_{\text{pml}} = O(12pn^3)$) and M2L. The memory cost of expansion coefficient for this 16 reduced SVD method is: $8N(2n^3 + 8p)/(7s)$.

5. NUMERICAL RESULTS. In the litterature, most of the theoretical comparisons between different FMMs make implicitly the assumption that there is no empty box in the octree. This comparison is not correct for surface interaction problems, because of a lot of empty boxes in the octree (see for example [6]). To take this fact into account, we use the theoretical comparison discussed in [11] and more precisely in [12]. This comparison is done at a level l_{max} fixed. Therefore, the cost of near interactions is the same for all methods and we are only interested on the cost of translation steps (steps from M2M to L2L or from PM2PM to PL2PL) and the cost at the finest step (S2M+L2T or S2PM+PL2T). For translation steps, the cost depends linearly on the average number of boxes in the interaction list α and so the comparison is done on α . For the finest step, the cost depend on evaluation of functions: Legendre polynomials or SH and so the comparison is done at fixed error. We introduce for method 1 and method 2, the maximum number of boxes of the interaction list for which method 1 is faster than method 2, denoted by α_{1-2} .

We use the error $\epsilon_{L^2}^2 = (\sum_{j=1}^N (\Phi_e(x_j) - \Phi_a(x_j))^2) / \sum_{j=1}^N \Phi_e(x_j)^2$, where $\Phi_e(x_j) = \sum_{y_i \in X^b} K(x_j, y_i) q_i$ for x_j in the box b , and Φ_a is an approximation with FMM. We consider here only the case of interactions of particules, of uniform intensity, distributed at the center of gravity of elements of surface mesh (quadrature points). The two example considered here are a sphere (radius 1) with 327680 elements and a cylinder (radius 1 and height 5) with 304200 elements.

Parameters. For a kernel $1/r^4$ and a precised error $\epsilon_{L^2} < 10^{-3}$ for the transfer step (M2L) and for a given box b ($y_i \in X_2^b$), we advocated to these parameters $n = 6, p = 50$ for 16 R-SVD, $L = 8$ for SH or RTCD and $L' = 11$ for Taylor in [11]. For the complete Multilevel fast multipole method (MLFMM) $n = 5, p = 25$ or $n = 6, p = 30$ are sufficient for 16 R-SVD for the far field error (see table 3). For the radiosity kernel, table 3 shows that these truncations give an error $\epsilon_{L^2} \leq 3.10^{-4}$ in the case of MLFMM algorithm.

Translation steps. Since we have no optimized software for a real numerical comparison of these methods, therefore, we compare theoretically the translation steps. Table 2 shows that the new method is always better than the SH method and better than the RTCD and Taylor for $\alpha > 12$ for $1/r^4$ (see [12] for more details on how this value is computed). We have to note at this point the version of Taylor proposed by Tausch is not considered [19]. As the FMM algorithm for the radiosity kernel is obtained using 16 times the algorithm for $1/r^4$, then the conclusion is the same for the radiosity equation.

Finest steps. In [12], we show that the cost of 16 R-SVD with $n = 6$ is equivalent to HS or RTCD with $L = 8$ and better than Taylor with $L' = 11$ for this error.

6. CONCLUSION. In this paper, we have investigated iterative methods and the FMM to accelerate the iterative solution of the radiosity equation: Taylor or SH expansions, the RTCD optimization and the reduced SVD method (R-SVD). The R-SVD, inspired by [5] and previously introduced in [11], is for a general kernel. In this paper, the algorithm of an optimized R-SVD for a kernel invariant

by translation satisfying some additional symmetries is presented: 16 R-SVD. A comparison of FMMs, with respect to the number of non-empty boxes in the interaction lists, is presented for the radiosity equation based on a FMM expansion for $1/r^4$. This comparison leads to the result that the 16 R-SVD is faster than the one of [8] and also in some configurations of the octree than the RTCD and Taylor FMMs. In the future, we plan to construct a fast iterative solver for the radiosity equation with these FMMs and to test the R-SVD on other forms of the radiosity kernel.

Table 2: Accuracy for different truncations for the 16 R-SVD for particles distributed at the center of gravity of elements of surface mesh on a sphere ($N = 327680$, $\alpha \simeq 44$) and a cylinder ($N = 304200$, $36 < \alpha < 45$).

case	$n = 5$	$n = 6$	$n = 6$
	$p = 25$	$p = 30$	$p = 50$
Sphere $1/r^4$	2.86 e-4	7.70 e-5	6.45 e-5
Cylinder $1/r^4$	3.96 e-4	1.65 e-4	1.63 e-5
Sphere radiosity	1.88 e-4	8.25 e-5	2.47 e-5
Cylinder radiosity	2.66 e-4	9.51 e-5	4.63 e-5

Table 3: value of α_{1-2} where the method 2 is the 16 R-SVD.

16 R-SVD	$n = 5$	$n = 6$	$n = 6$
	$p = 25$	$p = 30$	$p = 50$
Taylor ($L' = 11$)	4	10	24
SH ($L = 8$)	0	0	1
RTCD ($L = 8$)	5	12	27

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