Localisation analysis by BEM in damage mechanics

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Abstract. In this paper a thermodynamically consistent damage mechanic model is presented in the context of a boundary element formulation. In particular, the damage model of Lemaitre is considered. The boundary element method (BEM) is applied by introducing convenient inelastic strains which account for the irreversibly degeneration of the mechanical properties due to a diffused microcracking in the structure. The theoretical background of the model as well as the boundary element formulation are presented. The governing relations are first derived by the free energy potential fully complying with thermodynamic principles, then the flaw laws are obtained by assuming the existence of a damage activation function and under the hypothesis of generalised associative damage behaviour, finally numerical results are obtained by coupling suitably the BEM with the arclength methods.

Introduction

Quasibrittle materials, such as concrete, rock, tough ceramics, are characterised by the development of nonlinear fracture process zones, which can be macroscopically described as regions of highly localised strains. Continuum-based modelling of the progressive growth of microcracks and their coalescence requires constitutive laws with strain softening. Damage models are known to describe the accumulation of continuum damage, the initiation of micro cracks, and their coalescence to macro cracks as a function of the loading process. One of the most commonly used damage models, more phenomenologically based, was first introduced by Kachanov. An additional damage variable is introduced in the constitutive equations in order to describe the portion of the structure which is still able to carry load. After this original work, an impressive research activity has been carried out aimed both at the development of models suitable for describing a large number of materials and at solving the problems connected with softening behaviour, non-unique solutions and unstable paths.

In this paper the BEM is applied to the model initially proposed by Kachanov and then developed by [1]. Numerical results are shown in order to discuss the post-peak behaviour of such model. In the context of standard continuum damage mechanics, infact, softening leads to serious mathematical and numerical difficulties. The boundary value problem becomes ill-posed, and the numerical solution exhibits a pathological sensitivity to the computational discretisation. As a remedy, regularisation techniques enforcing a mesh-independent profile of localised strains are used. The governing relations of the damage model are derived by introducing a free energy potential which fully complies with thermodynamic principles; the state laws are written on the basis of the intrinsic dissipation at a given point and finally the flaw laws are obtained by assuming the existence of a damage activation function and under the hypothesis of generalised associative damage behaviour.

A similar boundary integral formulation, as obtained by [2], can be derived by introducing an additional domain integral which represents the nonlinear damage behaviour of the material. For softening localisation phenomena, it must be underlined that the nonlinear term tends to localise into a small region of the solid, therefore the advantage of BEM in reducing the number of unknowns by discretising only the boundary and the subregion in which the nonlinearity occurs is still valid. The BEM/arclength procedure for general material nonlinear problems presented in [3] is applied. The boundary element initial stress approach is coupled with the arclength constraint in order to deal with possible snap-back behaviours.

Numerical solutions are presented both for linear and for quadratic damage.

The damage model

In what follows a simple damage model is described. Damage may be interpreted at the microscale as the creation of microsurfaces of discontinuities: breaking of atomic bonds and plastic enlargement of microcavities. At the mesoscale, the number of broken bonds or the pattern of microcavities may be approximated in any plane by the area of the intersections of all the flaws with that plane. This area is scaled by the size of the representative volume element (RVE). The damage can be quantified by means of a scalar parameter d which, in any point, given the direction of a plane, represents the ratio between the effective area of the intersection of all microcracks lying in the plane and the area of the intersection of the plane with the RVE. The damage is assumed to be isotropic, i.e. d is assumed not to vary with the direction of the plane. The formulation is confined to the case of small induced strains. Let the Helmholtz free energy be of the form:

$$\Psi(\varepsilon, d, \xi) = \frac{1}{2}\varepsilon : f(d)\mathbf{C}^e : \varepsilon + \frac{1}{2} h\xi^2 \tag{1}$$

where the first term is the damage elastic strain energy and second term is the part of energy stored in the micro-structure related to the change of the material internal properties. ξ is a scalar kinematic internal variable which describes the damage hardening state. The dissipative irreversible mechanism associated to the damage is governed by the couple of variables $\dot{d}$ and $\dot{\xi}$ which appear in the expression of the intrinsic dissipation:

$$\dot{D} = \sigma : \dot{\varepsilon} - \dot{\Psi} \geq 0 \tag{2}$$

The expression of the free energy rate can be obtained by differentiating equation (1):

$$\dot{\Psi}(\varepsilon, d, \xi) = \varepsilon : f(d)\mathbf{C}^e : \dot{\varepsilon} + \frac{1}{2}\varepsilon : \frac{\partial f(d)}{\partial d}\mathbf{C}^e : \varepsilon\, \dot{d} + h\xi\dot{\xi} \tag{3}$$

By introducing this relation into (2), and reminding the inequality must hold for any admissible deformation mechanism, either nondissipative elastic or irreversible damaging one, the following state laws are obtained:

$$\sigma = f(d)\mathbf{C}^e : \varepsilon \tag{5a}$$

$$Y := -\frac{1}{2}\frac{\partial f(d)}{\partial d}\varepsilon : \mathbf{C}^e : \varepsilon \tag{5b}$$

$$\chi := h\xi \tag{5c}$$

The existence of a damage activation function $\phi(Y, \chi)$ is now assumed. Under the hypothesis of generalised associative damage behaviour, the damage activation function can be written in the following form:

$$\phi(Y, \chi) = Y - Y_0 - \chi \leq 0 \qquad \dot{\lambda} \geq 0 \qquad \dot{\lambda}\phi = 0 \quad \text{in}\,\Omega \tag{6}$$

where $\dot{\lambda}$ is the damage multiplier. Consequently the flaw laws read:

$$\dot{d} = \frac{\partial \phi}{\partial Y}\dot{\lambda} = \dot{\lambda}, \qquad \dot{\xi} = -\frac{\partial \phi}{\partial \chi}\dot{\lambda} = \dot{\lambda} \quad \text{in}\,\Omega \tag{7}$$

Now the material response to an assigned strain rate field $\dot{\varepsilon}(\mathbf{x})$ is investigated. If Ω_d is the part of the body which is damaged, the response at $\mathbf{x}$ is locally elastic if $\mathbf{x} \notin \Omega_d$, whereas at the points $\mathbf{x} \in \Omega_d$ the response is elastic-damaging and the following relations must hold:

$$\dot{\phi}(Y,\chi) \leq 0 \qquad \dot{d} = \dot{\xi} = \dot{\lambda} \geq 0 \qquad \dot{\phi}\dot{\lambda}_d = 0 \quad \text{in}\,\Omega_\mathrm{d} \tag{8}$$

Expanding the damage activation function in its rate form leads to:

$$\dot{\phi}(Y,\chi) = \frac{\partial \phi}{\partial Y}\dot{Y} + \frac{\partial \phi}{\partial \chi}\dot{\chi} = \dot{Y} - \dot{\chi} \leq 0 \qquad \dot{\lambda} = \dot{d} = \dot{\xi} \geq 0 \qquad \dot{\phi}\dot{\lambda} = 0 \tag{9}$$

where:

$$\dot{Y} = -\frac{1}{2}\frac{\partial^2 f(d)}{\partial d^2}\varepsilon : \mathbf{C}^e : \varepsilon\, \dot{d} - \frac{\partial f(d)}{\partial d}\varepsilon : \mathbf{C}^e : \dot{\varepsilon} \qquad \dot{\chi} = h\dot{\xi} \tag{10}$$

By introducing relations (10) into relation (9) the complete incremental damage problem can be stated as follows:

$$\dot{\phi}(Y,\chi) = -\frac{1}{2}\frac{\partial^2 f(d)}{\partial d^2}\varepsilon : \mathbf{C}^e : \varepsilon\, \dot{d} - \frac{\partial f(d)}{\partial d}\varepsilon : \mathbf{C}^e : \dot{\varepsilon} - h\dot{\xi} \leq 0 \tag{11a}$$

$$\dot{\lambda} = \dot{d} = \dot{\xi} \qquad \dot{\phi}\dot{\lambda} = 0 \qquad \text{in } \Omega_d \tag{11b}$$

The BEM in damage analysis

Applications of BEM to localisation phenomena can be found in [4-6]. The main limit of these approaches is the incapability to fully cope with problems involving snap-back behaviours, typically related to the localisation of the strains. In this contribution the coupling procedure BEM/arclength for physically nonlinear problems proposed in [3] will be adopted. Without loss of generality, the explicit version of the initial stress approach will be implemented within the context of the small strain theory. In order to keep a unified notation, these equations will be presented in rate form. In the absence of body forces, the displacement boundary integral equation can be written as

$$c_{ij}(\xi)\dot{u}_j(\xi) + \int_\Gamma t^*_{ij}(\xi,\mathbf{x})\dot{u}_j(\mathbf{x})d\Gamma(\mathbf{x}) = \int_\Gamma u^*_{ij}(\xi,\mathbf{x})\dot{t}_j(\mathbf{x})d\Gamma(\mathbf{x}) + \int_{V_d} \varepsilon^*_{ijk}(\xi,\mathbf{x})\dot{\sigma}^d_{jk}(\mathbf{x})\, d\Omega(\mathbf{x}) \tag{12}$$

and the stress in any internal point is given by:

$$\dot{\sigma}_{ij}(\mathbf{X}) = \int_\Gamma U^*_{ijk}(\mathbf{X},\mathbf{x})\dot{t}_k(\mathbf{x})d\Gamma(\mathbf{x}) - \int_\Gamma T^*_{ijk}(\mathbf{X},\mathbf{x})\dot{u}_k(\mathbf{x})d\Gamma(\mathbf{x}) + \int_{V_d} \Xi^*_{ijkl}(\mathbf{X},\mathbf{x})\dot{\sigma}^d_{kl}(\mathbf{x})\, d\Omega(\mathbf{x}) + g_{ij}(\dot{\sigma}^d(\mathbf{X})) \tag{13}$$

where Γ is the boundary of the domain of volume Ω, $\dot{\mathbf{u}}$ and $\dot{\mathbf{t}}$ are, respectively, displacement and traction increment fields on the boundary, and the expression of the fundamental solution t^*_{ij}, u^*_{ij}, ε^*_{ijk}, T^*_{ijk}, U^*_{ijk}, Ξ^*_{ijkl} and g_{ij} are given for instance in [7]. In the equations (12-13) ξ and $\mathbf{x}$ belong to the boundary and are usually referred as source and field point, respectively, $\mathbf{X}$ is an internal point. The nonlinear term σ^d, appearing in the above equations, can be considered utterly equivalent to the plastic stress. In fact:

$$\sigma = f(d)\mathbf{C}^e : \varepsilon = \mathbf{C}^e : \varepsilon - \sigma^d = \sigma^{el} - \sigma^d \tag{14}$$

By discretising both the boundary Γ in continuous quadratic elements and the part of Ω in which the damage is expected to occur in quadratic quadrilateral internal cells, by applying the Dirichlet/Neumann boundary conditions and by collocation technique, the equation (12) can be re-written as:

$$\mathbf{A}\,\dot{\mathbf{x}} = \dot{\mathbf{f}} + \mathbf{Q}\dot{\sigma}^{d} \tag{15}$$

whereas the internal and boundary stresses can be collected in:

$$\dot{\sigma}^{el} = -\mathbf{A}'\dot{\mathbf{x}} + \dot{\mathbf{f}}' + (\mathbf{Q}' + \mathbf{E}' + \mathbf{I})\dot{\sigma}^{d} = -\mathbf{A}'\dot{\mathbf{x}} + \dot{\mathbf{f}}' + \overline{\mathbf{Q}}\,\dot{\sigma}^{d} \tag{16}$$

The stress rate in any boundary point is obtained by special relations depending on traction rates and numerical tangential derivative of the displacement.
The evolution problem for a finite time step Δt and for any given strain increment $\Delta\varepsilon = \varepsilon_{s+1} - \varepsilon_{s}$ is considered. The iterative incremental procedure is based on a special coupling of the equations (15-16) with the arclength constrain. Furthermore a return-mapping algorithm, i.e.:

$$\Delta d = \frac{Y_{s+1} - \phi_s}{h} \quad \text{linear damage} \qquad\qquad \Delta d = \frac{(1-d)Y_{s+1} - \phi_s}{h + Y_{s+1}} \quad \text{quadratic damage} \tag{17}$$

The increment of the nonlinear term σ^{d}, which is necessary at every step of the iterative procedure, can be easily obtained:

$$\Delta\sigma^{d} = d\,\mathbf{C}^{e} : \Delta\varepsilon + \Delta d\mathbf{C}^{e} : \varepsilon_{s} \qquad \text{linear damage} \tag{18a}$$

$$\Delta\sigma^{d} = 2d\,(\mathbf{C}^{e} : \Delta\varepsilon - \Delta d\,\mathbf{C}^{e} : \varepsilon) - d^{2}\,\mathbf{C}^{e} : \Delta\varepsilon + 2d\,\mathbf{C}^{e} : \varepsilon \qquad \text{quadratic damage} \tag{18b}$$

The arclength methods represent the only alternative to classical load/displacement control algorithms which is able to pass critical limit point (i.e. snap-back or snap-through). The main idea is to consider the load increment as a variable: a new equation requiring that the new equilibrium point is searched on an arc of fixed radius and the center set in the previous equilibrium state is added.
In [3] the arclength constraint is introduced into a typical BEM iterative procedure for physically nonlinear problems. The resulting nonlinear system of equations is discussed and solved. In the generic time step, such procedure furnishes the following additive corrections to be evaluated sequentially:

$$\delta\mathbf{x} = -\mathbf{A}^{-1}\mathbf{R}_{0} + \mathbf{A}^{-1}\,\mathbf{f}\,\delta\lambda = \delta\mathbf{x}^{I} + \delta\lambda\,\delta\mathbf{x}^{II} \tag{19a}$$

$$\delta\lambda = -\frac{\Delta\mathbf{x}_{o}^{T}\,\delta\mathbf{x}^{I}}{\Delta\mathbf{x}_{o}^{T}\,\delta\mathbf{x}^{II}} \tag{19b}$$

$$\delta\sigma^{el} = -R_{o}' - \mathbf{A}'\,\delta\mathbf{x} + \mathbf{f}'\,\delta\lambda \tag{19c}$$

The subscript o indicates old, i.e. $(\,)_n = (\,)_o + \delta(\,)$, whereas R_o and R_o' are related to the error of the previous iteration step.

Numerical analyses

In all the numerical examples here presented a rectangular plate $1000\text{x}1250\,\text{mm}^2$ in plane stress and loaded on the horizontal top line by a constant load $p = 2.0\,N/mm^2$ is considered. The following values of the material properties are used: $E = 20000\,N/mm^2$, $\nu = 0.30$, $Y_0 = 1.0E-04\,N/mm^2$ and $h = 8.0E-04\,N/mm^2$. Three different internal meshes are used to evaluate the domain integral: 4x5, 8x10 and 12x15. The load-displacement diagrams describe graphically the relationship between the vertical displacement of the central point of the top horizontal line and the load factor λ.
The first example has the purpose to check the accuracy of the proposed procedure. Both linear damage and quadratic damage present good results: the uniform stress

field is captured properly by the model. Figure 1 presents the results for the three meshes considered. It must be underlined that, in the case of linear damage, the instability appears for every mesh, i.e. a lack of convergence at $\lambda = 0.9$ in the post peak branch occurs.

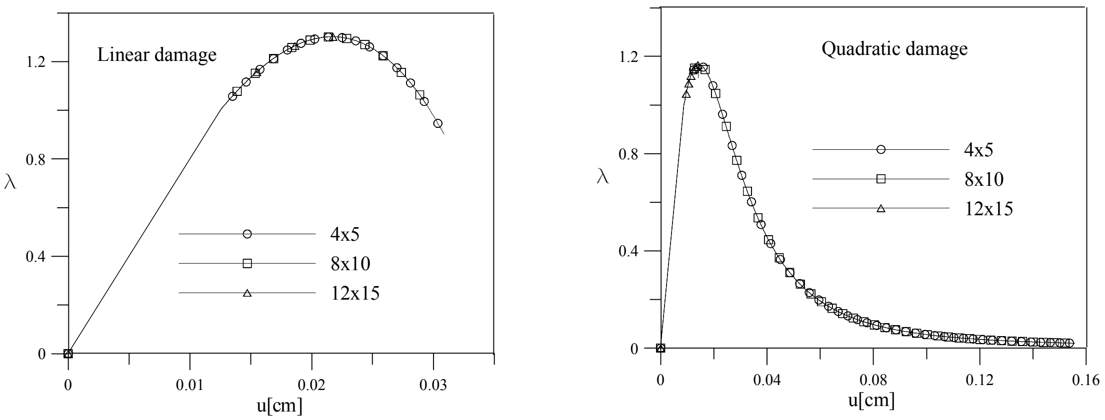

Fig. 1 – Load parameter versus top displacement – Uniform strength

On the other hand, in case of quadratic damage the load-displacement curve has no stop due to the lost of convergence, but the finest mesh cannot pass the peak point. Figure 2 shows the increasing of the damage parameter. Only in the case of quadratic damage the maximum value $d = 1$ is reached.

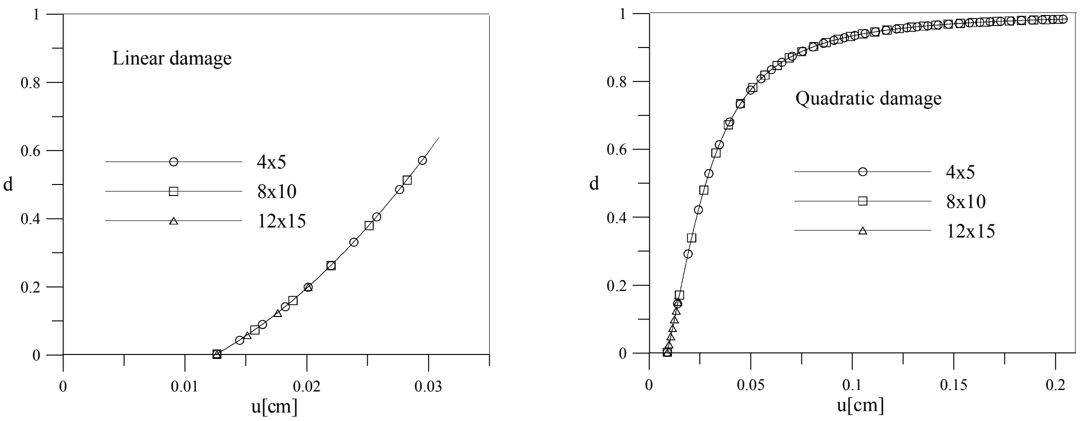

Fig. 2 – Damage parameter versus top displacement – Uniform strength

The second example is able to point out the constitutive instability due to the locality of the model. The initial value of Y_0 in the domain cell in the bottom-left corner of the plate is taken 5% lower than the value in the remaining domain cells. This is done in order to trigger the localisation of the damage and to show the pathological mesh dependence in the numerical results. Figure 3, infact, shows that the post-peak response is captured, with great numerical difficulties, only in the case of quadratic damage. Besides, both for linear and for quadratic damage, the numerical response is dependent on the internal mesh.

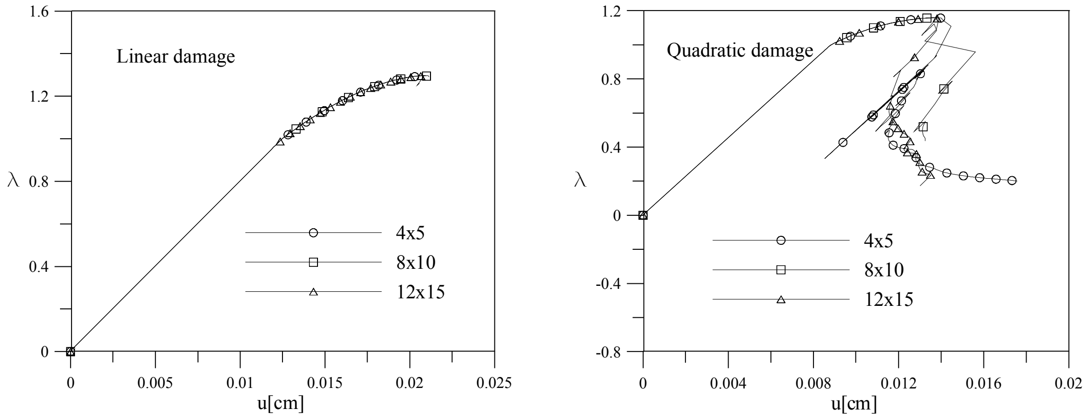

Fig. 3 – Load parameter versus top displacement – Weakened strength

Summary

A nonlinear Boundary Element procedure has been proposed for damage models. In order to pass critical limit points, the algorithm has been combined with the arclenght methods. Constitutive instability has been highlit in two simple numerical examples. The strain softening is source of theoretical and computational difficulties for continuum based structural modelling, which are substantially originated by constitutive instability matters as pointed out by many researchers. A non-local BEM formulation of integral type is developed in [5], whereas a gradient approach with an implicit boundary element formulation is proposed in [6]. Both papers deal with the classical plasticity. Numerical analyses are still in progress: some more work need to be done in order to implement regularised damage models and to obtain numerical results in more complex situations.

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