Three-dimensional Time-Independent Green's Functions for Unsaturated Soils

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Abstract. The *three-dimensional* Green's functions of the governing differential equations of unsaturated soils have been presented in this paper. The governing equations consist of equilibrium, air and moisture mass conservation and transfer equations. The Green's functions have been derived for *three-dimensional* deformable porous media with linear-elastic behavior for soil skeleton in a symmetric spherical domain. Some of the special unsaturated soil properties like *suction effect* and *dissolved air in water* have been considered. The results have been drawn and verified by comparing with *elastostatic* corresponding solutions.

Introduction

Among the numerical methods for solving the governing differential equations of the soils phenomena, the boundary element method is of the great importance due to its advantages. The capability of modeling infinite boundaries, no need for definition of interior mesh of element, and no unknowns associated with interior points of the domain in a numerical implementation, less data preparation time and less required computer time and storage for the same level of accuracy have resulted in a unique interesting numerical method.

The development of the boundary element method has been restricted by the necessity of deriving the Green's functions of the governing differential equations as a mathematical problem and in this regard the Green's function is one of the interesting topics in the engineering mathematics. The Green's functions have been presented for various sets of governing differential equations in exact and approximate forms. For the elastostatic equations the fundamental solutions have been derived by classical methods [1]. These Green's functions have been all derived for fluid saturated soils [2,3]. Although the most of the difficulties arises in deriving the Green's function for time dependent problems, they have not been presented for unsaturated case even for time independent problems. Some of the main difficulties between the formulation of saturated and unsaturated soils are one more equation and parameter (air pressure) , suction effects and dissolved air in water.

In unsaturated soils the differential equations are different from those of saturated case due to the presence of one more parameter (air pressure) and one more equation in one hand and the presence of suction and dissolved air in water effects in the other hand. This research is an attempt to derive such time-independent Green's functions for a *three-dimensional* axisymmetric domain in spherical coordinates as the two-dimensional case which have been presented in the previous paper.

Governing differential equations

The governing differential equations using the effective stress concept consist of [4]:
Solid skeleton. Equilibrium and (linear-elastic) constitutive equations for soil's solid skeleton including suction effects:

$$\left(\sigma_{ij} - \delta_{ij} p_a\right)_{,j} + p_{a,i} + b_i = 0 \tag{1}$$

$$d\left(\sigma_{ij} - \delta_{ij} p_a\right) = D\, d\varepsilon + D_s\left(dp_a - dp_w\right) \tag{2}$$

considering the strain-deformation relations:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{3}$$

One can conclude:

$$(\lambda + \mu)\, u_{j,ij} + \mu u_{i,jj} + (D_s - 1) p_{a,i} - D_s p_{w,i} + b_i = 0 \tag{4}$$

where λ and μ are Lamé's coefficients of soil elasticity and D_s is the coefficient of deformations due to suction effect. Also σ, ε, u_i, p_a, p_w and b_i stand for stress, strain, soil's dispacement in direction i, air and water pressure and body force in direction i, respectively.

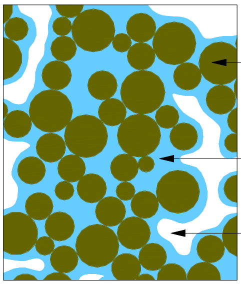

Fig. 1: Unsaturated soil scheme

Air phase. Time-independent continuity and transfer equations for the air phase, considering dissolved air in water:

$$div\left[\rho_a \left(u_a + H u_w\right)\right] = 0 \tag{5}$$

$$u_a = -K_a \nabla \left(\frac{p_a}{\gamma_a} + Z\right) \tag{6}$$

u_a, u_w, H, ρ_a and γ_a are air and water velocity, Henry's coefficient for dissolved air in water and density and specific weight of air, respectively. The air coefficient of permeability is defined as:

$$K_a = D \frac{\gamma_a}{\mu_a} \left[e\left(1 - S_r\right)\right]^E \tag{7}$$

in which μ_a, e and S_r are air dynamic viscosity, void ratio and degree of saturation, respectively and D and E are constants [5].

As the two-dimensional case it seems reasonable to dispense of the variations of K_a due to S_r and consequently of $(p_a - p_w)$ for the simplicity, since deriving the considered Green's functions will become too difficult, at least with common methods, which may be applied only to the linear differential equations. Therefore, the effects of S_r have been considered in air coefficient of permeability by assuming K_a as a multilinear function of $(p_a - p_w)$ for each finite domain. However eq (5) may be simplified as:

$$\frac{\rho_a K_a}{\gamma_a} \nabla^2 p_a + \frac{H \rho_a K_w}{\gamma_w} \nabla^2 p_w = 0 \tag{8}$$

Water phase. Time-independent continuity and transfer equations for the water phase:

$$div\left[\rho_w u_w\right] = 0 \tag{9}$$

$$u_w = -K_w \nabla \left(\frac{p_w}{\gamma_w} + Z\right) \tag{10}$$

where the water coefficient of permeability is defined as:

$$K_w = K_{wz0}\left(\frac{S_r - S_{ru}}{1 - S_{ru}}\right)^{3.5} \tag{11}$$

K_{wz0} is the intrinsic coefficient of permeability and S_{ru} is the residual degree of saturation. The same discussion as said for K_a shows that it is inevitable to dispense of the variations of K_w in finite domains of S_r. Therefore:

$$\frac{\rho_w K_w}{\gamma_w}\nabla^2 p_w = 0 \tag{12}$$

In fact, we simplified the governing differential equations to a set of linear system to make possible to derive the acceptable Green’s functions, while the main features of the unsaturated soils like suction effects and dissolved air in water have been kept in consideration. The governing differential equations (4,8,12) in the matrix form may be written as:

$$\left[C_{ij}\right]\times \vec{u} = \vec{f} \tag{13}$$

where $C_{ij} = c_{ij} d_{ij}$ and:

$$\vec{u} = u_i \qquad \vec{u}_4 = p_a \qquad \vec{u}_5 = p_w \qquad i = \overline{1,3} \tag{14}$$

$$\vec{f} = -b_i \qquad \vec{f}_4 = 0 \qquad \vec{f}_5 = 0 \qquad i = \overline{1,3} \tag{15}$$

$$c_{11} = \lambda + \mu \qquad c_{12} = \mu \qquad c_{13} = -1 + D_s \qquad c_{14} = -D_s$$

$$c_{21} = -\frac{\rho_a K_a}{\gamma_a} \qquad c_{22} = -\frac{H\rho_a K_w}{\gamma_w} \qquad c_{31} = -\frac{\rho_w K_w}{\gamma_w} \tag{16}$$

d_{ij} are the differential operators.

Green’s functions

Based on the method of *Kupradze* [6] which is a straightforward mathematical method, the Green’s functions of a set of differential equations with linear differential operators are the cofactors of C_{ij}:

$$\left[g_{ij}\right] = \left[C^*_{ij}\right]\varphi \tag{17}$$

in which φ is a potential function and satisfies the equation:

$$det\left(C_{ij}\right)\varphi + \delta(x) = 0 \tag{18}$$

in which $\delta(x)$ is the Dirac delta function in *three-dimensional* space. By definition of the potential function φ, a set of fundamental solutions will be achieved. This leads to such equation:

$$D\,\nabla^{10}\,\varphi + \delta(x) = 0 \qquad\qquad D = c_{12}^2\left(c_{11} + c_{12}\right)c_{21}c_{31} \tag{19}$$

where $\nabla^{2n} = \left(\nabla^2\right)^n$ is n occurrence of the Laplacian operator. The solution of eq (19) in an axisymmetric *three-dimensional* domain is:

$$\varphi = \frac{r^7}{161280 D\pi} \tag{20}$$

and g_{ij} or the Green’s functions are:

$$g_{ij} = \frac{(\lambda + 3\mu) r^2 \delta_{ij} + 2(\lambda + \mu) x_i x_j}{8\pi\, r^2 \mu\,(\lambda + 2\mu)}$$

$$g_{i3}=-\frac{\gamma_a x_i(1-D_s)}{8\pi r(\lambda+2\mu)K_a\rho_a}$$

$$g_{i4}=-\frac{x_i\left[D_s K_a\gamma_w-H(1-D_s)K_w\gamma_a\right]}{8\pi r(\lambda+2\mu)K_a K_w\rho_w}$$

$$g_{33}=-\frac{\gamma_a}{4\pi r K_a\rho_a}\qquad g_{44}=-\frac{\gamma_w}{4\pi r K_w\rho_w}$$

$$g_{34}=g_{43}=0\qquad g_{3i}=g_{4i}=0\qquad i,j=\overline{1,3}\tag{21}$$

which while H and D_s approach to zero, approach to elastostatic Green's functions [1,7]:

$$g_{ij}=\frac{(\lambda+3\mu)r^2\delta_{ij}+2(\lambda+\mu)x_i x_j}{8\pi r^2\mu(\lambda+2\mu)}$$

$$g_{i3}=-\frac{\gamma_a x_i}{8\pi r(\lambda+2\mu)K_a\rho_a}\qquad g_{i4}=0\qquad g_{3i}=g_{4i}=0$$

$$g_{33}=-\frac{\gamma_a}{4\pi r K_a\rho_a}\qquad g_{44}=-\frac{\gamma_w}{4\pi r K_w\rho_w}$$

$$g_{34}=g_{43}=0\qquad g_{3i}=g_{4i}=0\qquad i,j=\overline{1,3}\tag{22}$$

For instance, The derived Green's functions are shawn through Figs. 2 to 5 with the following initial values:

$$E=3\times10^4\ kPa\qquad \nu=0.35\qquad H=0.02\qquad D_s=2\qquad g=9.806\ m/s^2$$

$$\rho_a=1.293\ kg/m^3\qquad \rho_w=1000\ kg/m^3\qquad \mu_a=1.85\times10^{-5}\ kg/ms$$

$$a_{Kw}=1.2\times10^{-9}\ m/s\qquad \alpha_{Kw}=5\qquad S_r=0.5\qquad S_{ru}=0.05$$

$$D_{Ka}=1\times10^{-4}\ m^2\qquad E_{Ka}=2.6\qquad e_0=0.75\qquad z=1.00\ m\tag{23}$$

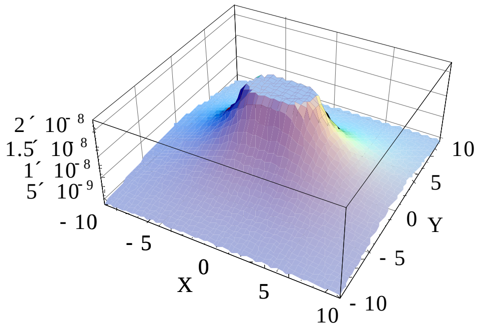

Fig. 2: Green's function g_{11}
Solid skeleton displacement in direction one due to *Hevisaide* point load in direction one.

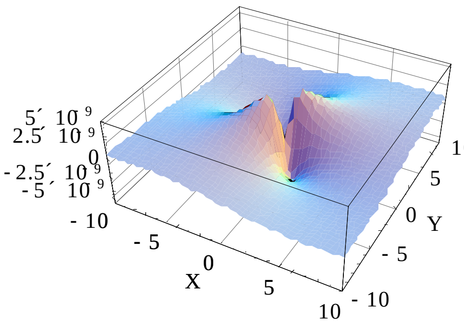

Fig. 3: Green's function g_{12}
Solid skeleton displacement in direction one due to *Hevisaide* point load in direction two.

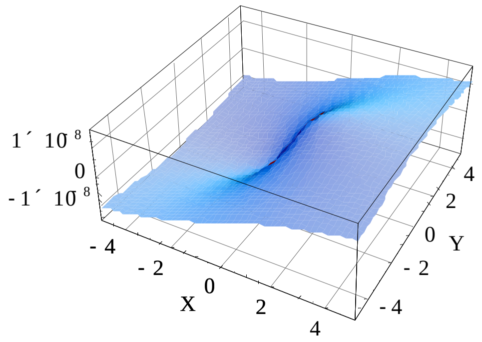

Fig. 4: Green’s function g_{14}
Solid skeleton displacement in direction one due to injection of air unit volume.

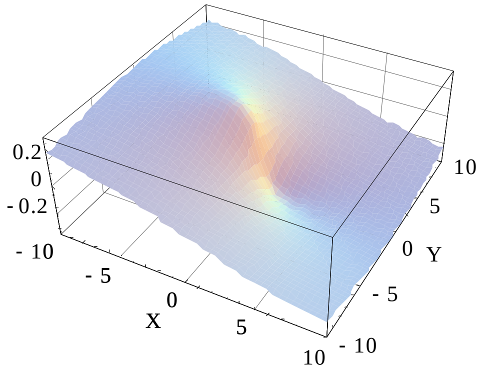

Fig. 5: Green’s function g_{15}
Solid skeleton displacement in direction one due to injection of water unit volume.

Conclusion

The closed form Green’s functions of *three-dimensional* governing differential equations of unsaturated soils, considering the *suction effects* and *dissolved air in water*, have been derived. For verification of the results, it has been demonstrated that if the conditions approach to *elastostatic* case, the Green’s functions will approach to elastostatic Green’s functions exactly. Although the mathematical procedure is straightforward, a set of fundamental solutions for the unsaturated case have been introduced as a new experience. The derived Green's functions may be used to develop a boundary element computer program for unsaturated soil's.

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