Two-dimensional Time-Independent Green's Functions for Unsaturated Soils

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Keywords: Green's function; Unsaturated soil; Boundary element method.

Abstract. In this article, after a brief discussion about the formulation of unsaturated soils including the equilibrium, air and moisture transfer equations, the closed form Green's functions of the governing differential equations for an unsaturated *two-dimensional* deformable porous media with linear elastic behavior for a symmetric polar domain, considering *suction effects* and *dissolved air in water*, have been introduced. The results have been compared with their corresponding *elastostatic* Green's functions.

Introduction

Numerical modeling has been largely developed in soil mechanics behavior by different methods. Among them, the development of the boundary element method, which is the most suitable one for domain with infinite boundary conditions like soils media, has been restricted by the necessity of deriving the Green's functions of the governing differential equations.

The Green's functions for elastostatic problems have been derived by classical methods. Particularly, the Green's functions for poroelastic and thermoelastic problems have been introduced in static and dynamic cases [1]. These Green's functions have been all derived for fluid saturated soils [2,3]. Although the most of the difficulties arises in deriving the Green's function for time-dependent problems, they have not been presented yet for unsaturated case even for time independent problems.

In unsaturated soils the differential equations are different from those of saturated case due to the presence of one more parameter (air pressure) and one more equation in one hand and the presence of suction and dissolved air in water effects in the other hand. This research is an attempt to derive such time-independent Green's functions for a *two-dimensional* axisymmetric domain in polar coordinates and specially consider the mentioned effects. In the following paper the corresponding Green's functions will be presented for the three-dimensional case.

Governing differential equations

The governing differential equations using the effective stress concept consist of [4]:
Solid skeleton. Equilibrium and (linear-elastic) constitutive equations for soil's solid skeleton including suction effects:

$$\left(\sigma_{ij} - \delta_{ij} p_a\right)_{,j} + p_{a,i} + b_i = 0 \tag{1}$$

$$d\left(\sigma_{ij} - \delta_{ij} p_a\right) = D\, d\varepsilon + D_s\left(dp_a - dp_w\right) \tag{2}$$

considering the strain-deformation relations:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{3}$$

may be written as:

$$(\lambda + \mu)\, u_{j,ij} + \mu\, u_{i,jj} + (D_s - 1) p_{a,i} - D_s p_{w,i} + b_i = 0 \tag{4}$$

where λ and μ are Lamé's coefficients of soil elasticity and D_s is the coefficient of deformations due to suction effect. Also σ, ε, u_i, p_a, p_w and b_i stand for stress, strain, soil's dispacement in direction i, air and water pressure and body force in direction i, respectively.

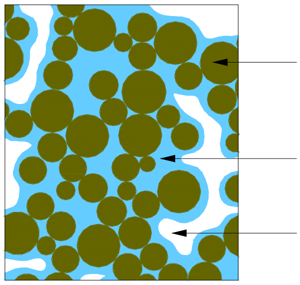

Fig. 1: Unsaturated soil scheme

Air phase. Time-independent continuity and transfer equations for the air phase, considering dissolved air in water:

$$div[\rho_a(u_a + Hu_w)] = 0 \quad (5)$$

$$u_a = -K_a \nabla\left(\frac{p_a}{\gamma_a} + Z\right) \quad (6)$$

u_a, u_w, H, ρ_a and γ_a are air and water velocity, Henry's coefficient for dissolved air in water and density and specific weight of air, respectively. The air coefficient of permeability is defined as:

$$K_a = D\frac{\gamma_a}{\mu_a}[e(1 - S_r)]^E \quad (7)$$

in which μ_a, e and S_r are air dynamic viscosity, void ratio and degree of saturation, respectively and D and E are constants [5].

It seems reasonable to dispense of the variations of K_a due to S_r and consequently of $(p_a - p_w)$ for the simplicity, since deriving the considered Green's functions will become too difficult, at least with common methods, due to the nonlinearity of the governing differential equations. Therefore, the effects of S_r have been considered in air coefficient of permeability by assuming K_a as a multilinear function of $(p_a - p_w)$ for each finite domain. However eq (5) may be written as:

$$\frac{\rho_a K_a}{\gamma_a}\nabla^2 p_a + \frac{H\rho_a K_w}{\gamma_w}\nabla^2 p_w = 0 \quad (8)$$

Water phase. Time-independent continuity and transfer equations for the water phase:

$$div[\rho_w u_w] = 0 \quad (9)$$

$$u_w = -K_w \nabla\left(\frac{p_w}{\gamma_w} + Z\right) \quad (10)$$

where the water coefficient of permeability is defined as:

$$K_w = K_{wz0}\left(\frac{S_r - S_{ru}}{1 - S_{ru}}\right)^{3.5} \quad (11)$$

K_{wz0} is the intrinsic coefficient of permeability and S_{ru} is the residual degree of saturation. A discussion similar to that made for K_a shows that it is inevitable to dispense of the variations of K_w in finite domains of S_r. Assuming constant K_w for the specified regions of S_r is, indeed, assuming it as a multilinear function of S_r which simply reflects the basic concept of the relation between K_w and S_r. Therefore:

$$\frac{\rho_w K_w}{\gamma_w}\nabla^2 p_w = 0 \tag{12}$$

The mentioned differential equations are simplified so that make possible to derive the acceptable Green's functions, while the main features of the unsaturated soils such as suction effects and dissolved air in water have been kept in consideration. One can arrange the governing differential equations (4,8,12) in the matrix form:

$$\left[C_{ij}\right]\times \vec{u} = \vec{f} \tag{13}$$

where $C_{ij} = c_{ij} d_{ij}$ and:

$$\vec{u} = u_i \qquad \vec{u}_3 = p_a \qquad \vec{u}_4 = p_w \qquad i = \overline{1,2} \tag{14}$$

$$\vec{f} = -b_i \qquad \vec{f}_3 = 0 \qquad \vec{f}_4 = 0 \qquad i = \overline{1,2} \tag{15}$$

$$c_{11} = \lambda + \mu \qquad c_{12} = \mu \qquad c_{13} = -1 + D_s \qquad c_{14} = -D_s$$

$$c_{21} = -\frac{\rho_a K_a}{\gamma_a} \qquad c_{22} = -\frac{H\rho_a K_w}{\gamma_w} \qquad c_{31} = -\frac{\rho_w K_w}{\gamma_w} \tag{16}$$

d_{ij} are the differential operators.

Green's functions

One of the few methods for deriving the Green's functions matrix of a system of differential equations is the method of *Kupradze* [6], which is a straightforward mathematical method. Based on this method the Green's functions are the cofactors of C_{ij} :

$$\left[g_{ij}\right] = \left[C_{ij}^*\right]\varphi \tag{17}$$

in which φ is a potential function and satisfies the equation:

$$det\left(C_{ij}\right)\varphi + \delta(x) = 0 \tag{18}$$

in which $\delta(x)$ is the Dirac delta function in *two-dimensional* domain. By definition of the potential function φ, a set of fundamental solutions will be achieved. This leads to such equation:

$$D\,\nabla^8 \varphi + \delta(x) = 0 \qquad D = c_{12}\left(c_{11} + c_{12}\right)c_{21}c_{31} \tag{19}$$

where $\nabla^{2n} = \left(\nabla^2\right)^n$ is n occurrence of the Laplacian operator. The solution of eq (19) in an axisymmetric domain is:

$$\varphi = \frac{r^6\left[5 - 6\,Log(r)\right]}{27648 D\pi} \tag{20}$$

and the g_{ij} or Green's functions are:

$$g_{ij} = \frac{\left[(\lambda+\mu) - 2(\lambda+3\mu)Log(r)\right]r^2\delta_{ij} + 2(\lambda+\mu)x_i x_j}{8\pi r^2 \mu\,(\lambda+2\mu)}$$

$$g_{i3} = \frac{\gamma_a x_i\left[1 + 2Log(r)\right](1 - D_s)}{8\pi(\lambda+2\mu)K_a\rho_a}$$

$$g_{i4} = \frac{x_i\left[1 + 2Log(r)\right]\left[D_s K_a \gamma_w - H(1 - D_s)K_w\gamma_a\right]}{8\pi(\lambda+2\mu)K_a K_w \rho_w}$$

$$g_{33} = \frac{\gamma_a Log(r)}{2\pi\, K_a\rho_a} \qquad g_{44} = \frac{\gamma_w Log(r)}{2\pi\, K_w\rho_w}$$

$$g_{34} = \tilde{g}_{43} = 0 \qquad g_{3i} = g_{4i} = 0 \qquad i,j = \overline{1,2} \tag{21}$$

It is evident that while H and D_s approach to zero, the Green's functions in eq (21) approach to elastostatic Green's functions [1,7]:

$$g_{ij} = \frac{[(\lambda+\mu) - 2(\lambda+3\mu)Log(r)]r^2\delta_{ij} + 2(\lambda+\mu)x_i x_j}{8\pi r^2 \mu(\lambda+2\mu)}$$

$$g_{i3} = \frac{\gamma_a x_i [1+2Log(r)]}{4\pi(\lambda+2\mu)K_a\rho_a} \qquad g_{i4} = 0 \qquad g_{3i} = g_{4i} = 0$$

$$g_{33} = \frac{\gamma_a Log(r)}{2\pi K_a \rho_a} \qquad g_{44} = \frac{\gamma_w Log(r)}{2\pi K_w \rho_w}$$

$$g_{34} = g_{43} = 0 \qquad g_{3i} = g_{4i} = 0 \qquad i,j = \overline{1,2} \tag{22}$$

For instance the derived Green's functions are drawn through Figs. 2 to 5 with the following parameters:

$$\begin{array}{llll} E = 3\times10^4\ kPa & \nu = 0.35 & H = 0.02 & D_s = 2 \\ \rho_a = 1.293\ kg/m^3 & \rho_w = 1000\ kg/m^3 & \mu_a = 1.85\times10^{-5}\ kg/ms & \\ a_{Kw} = 1.2\times10^{-9}\ m/s & \alpha_{Kw} = 5 & S_r = 0.5 & S_{ru} = 0.05 \\ D_{Ka} = 1\times10^{-4}\ m^2 & E_{Ka} = 2.6 & e_0 = 0.75 & g = 9.806\ m/s^2 \end{array} \tag{23}$$

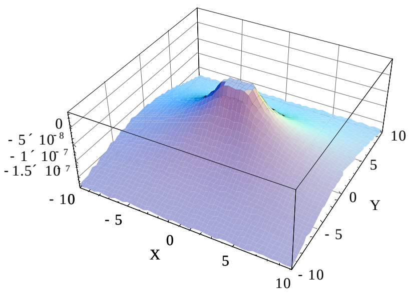

Fig. 2: Green's function g_{11}
Solid skeleton displacement in direction one due to *Hevisaide* point load in direction one.

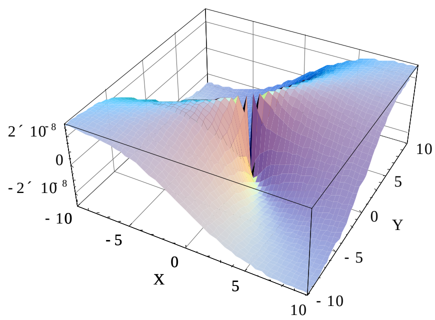

Fig. 3: Green's function g_{12}
Solid skeleton displacement in direction one due to *Hevisaide* point load in direction two.

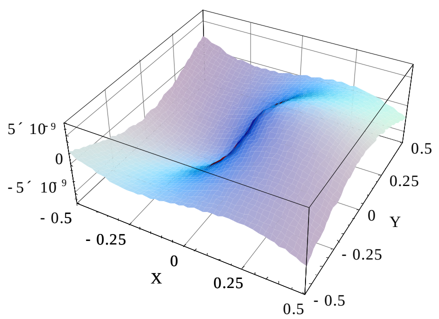

Fig. 4: Green's function g_{13}
Solid skeleton displacement in direction one due to injection of air unit volume.

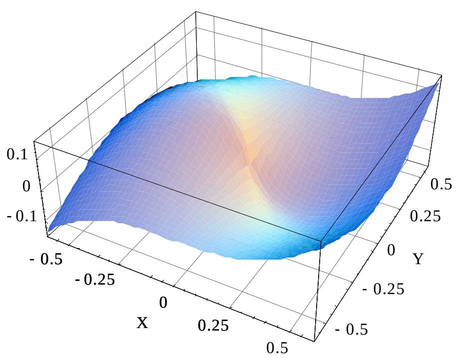

Fig. 5: Green's function g_{14}
Solid skeleton displacement in direction one due to injection of water unit volume.

Conclusion

In this research the closed form Green's functions of *two-dimensional* governing differential equations of unsaturated soils, including equilibrium equations with linear elastic constitutive equations and two equations of air and water transfer, considering *suction effects* and *dissolved air in water*, have been derived. For verification of the results, it has been demonstrated that if the conditions approach to elastostatic case, the Green's functions will approach to *elastostatic* Green's functions exactly. Although the mathematical procedure is not very complicated, it seems to be a new experience to introduce a set of fundamental solutions for the unsaturated case, probably for the first time. The derived Green's functions may be used to develop a boundary element computer program for modeling unsaturated soil's problems in steady state.

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