Section Seven

Heat and Fluid Flow

DRM APPLIED TO THE TIME-STEPPING BEM FOR TRANSIENT HEAT CONDUCTION

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Abstract This paper presents application of dual reciprocity method (DRM) to time-stepping BEM for the transient heat conduction problem in homogeneous media. The integral equation fomulation uses the fundamental solution of Laplace equation, and time derivative is approximated by the time-stepping method. Hence the domain integral arises in the boundary integral equation. This domain integral is transformed into boundary integrals by using a new set of radial basis functions for DRM. The details of the proposed DRBEM are presented, and a computer code is developed for two-dimensional problems. The usefulness of the method is demonstrated through numerical computation.

1 Introduction

This paper is concerned with a dual reciprocity boundary element method (DRBEM) [1, 2, 3] for solving the transient heat conduction problems in homogeneous media. The time-stepping scheme is employed to aproximate time derivative in the governing equation. The reduced differential equation thus obtained is transformed into the boundary integal equation in the usual manner using the fundamental solution of Laplace equation. The resulting boundary ingetral equation includes domain integrals related to the pseudo initial conditon.

In the authors' previous paper[4, 5], the domain integral is evaluated by introducing a cell division of the domain. The present paper aims at a more effective solution procedure based on dual reciprocity method. At this end, the dual reciprocity method can be applied to transforming the domain integral into boundary integrals. The authors have reported successful applications of DRM to the steady-state heat conduction problems in inhomogeneous and also the temperature-dependent materials[6, 7]. The idea of DRM could also be successfully applied to the present time-dependent problems.

The present paper describes in detail how to apply the DRM to the solution of the boundary ingetarl equation including a domain integral related to the pseudo initial conditon. A suitable radial basis function[3, 8] is introduced into the formalation. The details of the proposed DRBEM are presented, and a computer code is developed for two-dimensional problems. Finally, an example is computed by the computer code to demonstrate the usefulness and versality of the proposed method of solution.

2 Governing Differential Equation

The present paper is concerned with the unsteady-state heat conduction problem in a homogeneous medium. Under the assumption of no heat source, the governing differential

equation can be expressed by

$$\nabla^2 u(x,t) = \frac{1}{k}\frac{\partial u(x,t)}{\partial t} \tag{1}$$

where u is the temperature, λ the thermal conductivity and ∇ the gradient operator.

The boundary conditions and initial condition are expressed as follows:

$$\begin{aligned} &u(x,t) = \bar{u}, && \in \Gamma_u \\ &q(x,t) = \frac{\partial u(x,t)}{\partial n} = \bar{q}, && \in \Gamma_q \\ &u(x,0) = u_0, \end{aligned} \tag{2}$$

where n is the outward unit normal to the boundary Γ, and $\partial u(x,t)/\partial n$ the normal derivative. Γ_u and Γ_q stand for the boundary portions which are subject to the prescribed-temperature and prescribed-heat flux conditions, respectively. In addition, $\bar{u}$ and $\bar{q}$ are prescribed quantities of u and q on the boundary, while u_0 is the initial temperature.

3 Integral Equation Formulation

In order to approximate the time-derivative of the governing differential equation, we introduce the time-stepping approximation[4, 5]. The reduced differential equation can be expressed in the following form:

$$\nabla^2 u(x,t) = \frac{1}{k}\big\{au(x,t) - a_0 u_0(x, t - \Delta t)\big\} \tag{3}$$

where a and a_0 are the coefficients for the time-steping approximation, while $u_0(x, t-\Delta t)$ denotes the temperature data corresponding to the previous time steps.

For the integral equation formulation of this problem, we shall use the fundamental solution of the Laplace equation, which can be given for two-dimensional problems by

$$u^*(x,y) = \frac{1}{2\pi}\ln\left(\frac{1}{r}\right) \tag{4}$$

where r denotes the diatance between the source point y and a field point x.

Multiplying both the sides of Eq. (3) with the above fundamental solution, we integrate it over the whole spatial domain. Integrating it by parts repeatedly, we can derive the following boundary integral equation:

$$\begin{aligned} c(y)u(y,t) &+ \int_\Gamma q^*(x,y)u(x,t)d\Gamma - \int_\Gamma u^*(x,y)q(x,t)d\Gamma \\ &= -\int_\Omega \frac{u^*(x,y)}{k}\big\{au(x,t) - a_0 u_0(x,t-\Delta t)\big\}d\Omega \end{aligned} \tag{5}$$

where $q^*(x,y)$ stands for the normal flux of the fundamental solution $u^*(x,y)$.

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4 Application of DRM

First, we shall approximate the right-hand side of Eq. (3) in the following way:

$$\frac{1}{k}\{au(x,t) - a_0 u_0(x, t - \Delta t)\} = \sum_{\ell=1}^{N+L} \alpha^\ell f(x, z^\ell) \tag{6}$$

where N is the number of nodal points on the boundary, whereas L is the number of collocation points in the inner domain. We also denote by α^ℓ, $f(x, z^\ell)$ and z^ℓ, respectively, the unknown coefficients, the approximate function and a collocation point of DRM.

Now, we introcude a particular solution $\hat{u}(x, z^\ell)$ which satisfies the following equation:

$$\nabla^2 \hat{u}(x, z^\ell) = f(x, z^\ell) \tag{7}$$

If we use this particular solution $\hat{u}(x, z^\ell)$, the time-stepping approxiation of Eq. (3) can be expressed in terms of particular solutions and the unknown coefficents α^ℓ. We then obtain

$$\nabla^2 u(x,t) = \sum_{\ell=1}^{N+L} \alpha^\ell \nabla^2 \hat{u}(x, z^\ell) \tag{8}$$

Multiplying the above equation with the Laplace fundamental solution, we integrate it over the whole spatial domain. Then, integrating it by parts repeatedly, we can arrive at the following boundary integral equation:

$$\begin{aligned} &c(y)u(y,t) + \int_\Gamma q^*(x,y)u(x,t)d\Gamma \\ &- \int_\Gamma u^*(x,y)q(x,t)\Gamma = \sum_{\ell=1}^{N+L} \alpha^\ell \Big\{ c(y)\hat{u}(y,t) \\ &+ \int_\Gamma q^*(x,y)\hat{u}(x,z^\ell)d\Gamma - \int_\Gamma u^*(x,y)\hat{q}(x,z^\ell)d\Gamma \Big\} \end{aligned} \tag{9}$$

where $\hat{q}(x, z^\ell)$ denotes the normal flux of a particular solution $\hat{u}(x, z^\ell)$.

As has been well known [1, 2, 3], a variety of the functions can be applied to the formulation. We shall use in this study the following function:

$$f(x, z^\ell) = (1 - r^\ell/a_s)^3 (3r^\ell/a_s + 1) \tag{10}$$

where a_s is the support radius.

Knowing the two points x and z^ℓ, we can calculate the approximate function, particular solution and its normal derivative. Furthermore, applying the boundary integral equation (9) to all the collocation points, we can eventually obtain the system of $L + N$ equations which is summarized in the following matrix form:

$$\boldsymbol{Hu} - \boldsymbol{Gq} = \Big[\boldsymbol{H}\hat{\boldsymbol{U}} - \boldsymbol{G}\hat{\boldsymbol{Q}}\Big]\boldsymbol{\alpha} \tag{11}$$

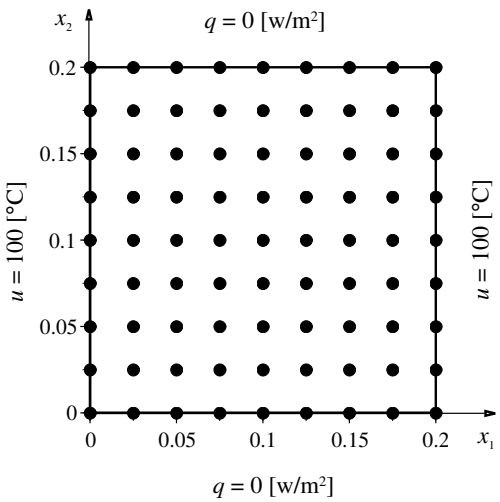

Figure 1 Analysis model and discretization

Applying Eq. (6) to all the collocation points, $\boldsymbol{\alpha}$ can be expressed in terms of $\boldsymbol{u}$ and $\boldsymbol{u}_0$. We eventually obtain

$$\boldsymbol{Hu} - \boldsymbol{Gq} = \boldsymbol{Ru} - \boldsymbol{U}_0 \tag{12}$$

in which

$$\boldsymbol{R} = \frac{a}{k}\left[\boldsymbol{H}\hat{\boldsymbol{U}} - \boldsymbol{G}\hat{\boldsymbol{Q}}\right]\boldsymbol{F}^{-1} \tag{13}$$

$$\boldsymbol{U}_0 = \frac{a_0}{k}\left[\boldsymbol{H}\hat{\boldsymbol{U}} - \boldsymbol{G}\hat{\boldsymbol{Q}}\right]\boldsymbol{F}^{-1}\boldsymbol{u}_0 \tag{14}$$

If we solve Eq. (12) for the unknowns by applying the given boundary conditions, we can obtain all the nodal values on the boundary as well as in the inner domain at the time-step under consideration.

5 Numerical Computation and Discussion

To demonstrate the usefulness of the present DRBEM, we now apply it to an example, and discuss the results obtained. The square region shown in Fig. 1 is analyzed.

The boundary is discretized by using 16 quadratic bounadry elements ; totally 32 nodes on the boundary. 49 collocation points are uniformly located in the inner domainn as shown in Fig. 1. In this study, we assume the support radius as $a_s = 1$.

It is assumed that the sides $x_1 = 0$ and $x_1 = 0.2$ are subject to the Dirichlet condition $u = 100$ [°C], while the sides $x_2 = 0$ and $x_2 = 0.2$ are subject to the Neumann condition $q = 0$. The initial condition is assumed such that the temperature in the whole region is

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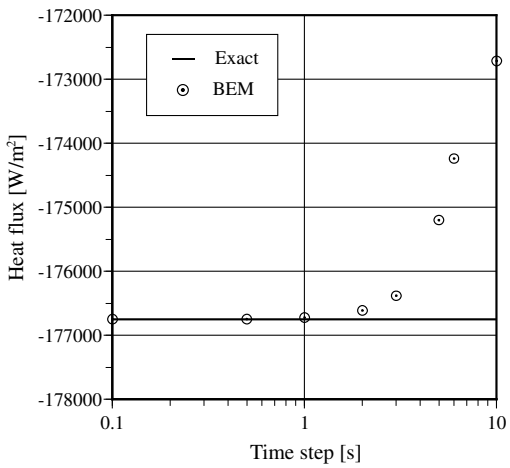

Figure 2 Heat flux on the origin after 30 [s]

0 [°C]. In addition, it is assumed that the heat conduction coefficient λ=120 $[\mathrm{W/mK}]$, mass density ρ=2800 $[\mathrm{kg/m^3}]$, and the specific heat $c = 880$ $[\mathrm{J/kgK}]$.

In this study we have employed the time-stepping approximation for the time derivative, and hence attention should be paid to an appropriate width of time-step. To this end, computation is performed for a number of different time-steps. Figure 2 summarizes the results obtained: It shows the heat flux at the point $x_1 = x_2 = 0$ after 30 seconds, obtained by the present method. It can be seen that a time-step width smaller than 0.5 [s] provides accurate numerical results. It is interesting to note that a negligibly small influence of the time-step width appears on the temperature.

In Fig. 3 are shown the results obtained for the temperature variations in time on the axis x_1, compared with the exact solution. It can be concluded that the present DRBEM based on the time-stepping method can provide very accurate numerical results.

6 Conclusions

The dual reciprocity boundary element method has been developed for the unsteady heat conduction problem in homogeneous media. The time derivative was approximated by the time-stepping scheme. The integral equation formulation was presented and its numerical implementation was made. An example was computed by using the computer code deveoped. The numerical results were compared with the analytical solutions, whereby the usefulness of the present method of solution was demonstrated.

As future research work, it can be recommended to apply the proposed method to the nonlinear problems with temperature-dependent materials.

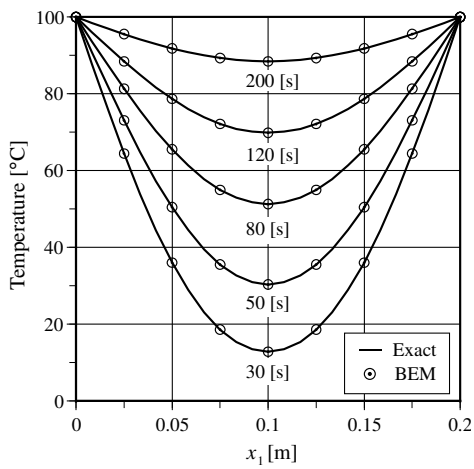

Figure 3 Results of temperature along axis x_1

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A Hypersingular Boundary Integral Formulation for Heat Conduction Across an Imperfect Interface

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Abstract. A hypersingular boundary integral formulation is derived for a steady-state two-dimensional heat conduction problem involving a bimaterial with a microscopically imperfect interface. To describe the interfacial condition, a macroscopic model which allows for a temperature jump which is proportional in magnitude to the thermal heat flux at the interface is used. For a specific problem, the integral equations in the formulation are solved numerically to determine the temperature field in the bimaterial.

Introduction

Composites which are made up of two or more dissimilar materials play an important role in modern technology. In many studies, the dissimilar materials are assumed to be perfectly joined or bonded to one another along their common boundaries (see e.g. Ang [1], Berger and Karageorghis [2], Clements [3] and Lee and Kim [4]).

However, a perfect bond does not exist in reality, as microscopic imperfections are bound to be present along the interfaces of the materials. Thus, in recent years, there is a growing interest among researchers in the investigation of microscopically imperfect interfaces in layered and composite materials (see e.g. Benveniste and Miloh [5], Fan and Sze [6], Benveniste [7], Torquato and Rintoul [8] and other references therein). In heat conduction problems, a macroscopic model for studying such an imperfect interface allows for a temperature jump which is proportional in magnitude to the thermal heat flux at the interface.

In the present paper, the two-dimensional problem of determining the steady-state temperature distribution in a thermally isotropic bimaterial with a homogeneously imperfect planar interface is considered. On the

exterior boundary of the bimaterial, either the temperature or the heat flux (not both) is known at each and every point on the exterior boundary of the bimaterial.

The temperature is expressed in terms of a boundary integral over the exterior boundary of the bimaterial and the imperfect planar interface. With the use of a suitable Green's function for the corresponding perfect interface, the only unknown function that appears in the integral over the imperfect interface is the temperature jump. The boundary integral expression for the temperature can be suitably differentiated to obtain the heat flux for formulating the condition on the imperfect interface. This gives rise to hypersingular boundary integral equations for the problem under consideration. The proposed approach is similar to that of Chen and Hong [9] for solving a heat conduction problem with a degenerate boundary. For a specific problem, the equations are solved numerically in order to determine the temperature field in the bimaterial.

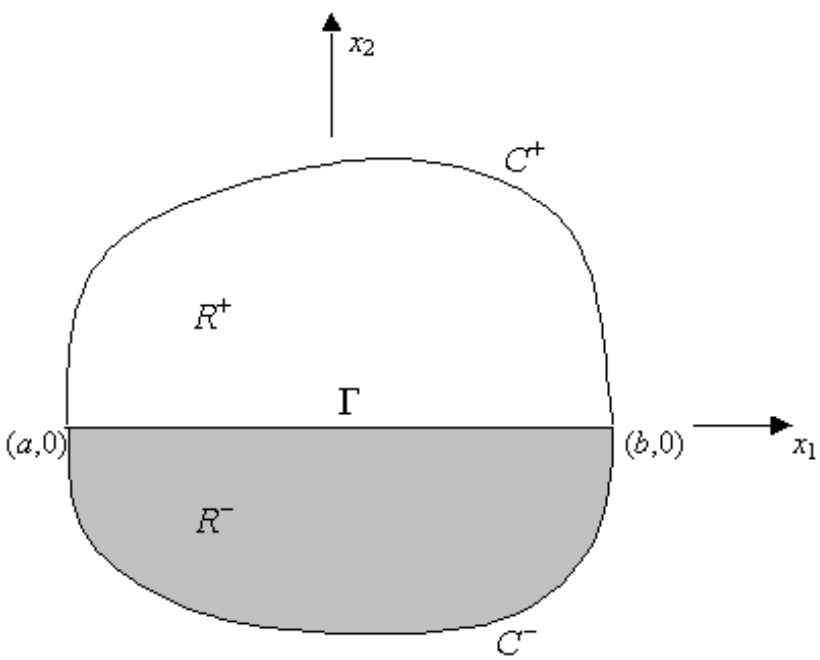

Figure 1. A sketch of the geometry of the problem.

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The Problem

Referring to an $0x_1x_2x_3$ Cartesian co-ordinate system, consider a body which comprises two homogeneous materials with possibly different thermal properties. The geometry of the body is independent of the x_3 co-ordinate. On the $0x_1x_2$ plane, the interface separating the two materials is the straight line segment Γ which lies on part of the x_1-axis between the points $(a, 0)$ and $(b, 0)$ (where a and b are given real numbers such that $a < b$), while the exterior boundary of the body is the simple closed curve C. The curve C consists of two parts: C^+ which lies above the x_1-axis, and C^- below the axis. A sketch of the geometry is given in Figure 1. The regions enclosed by $C^+ \cup \Gamma$ and $C^- \cup \Gamma$ are denoted by R^+ and R^- respectively.

If the steady-state temperature field in the body is independent of x_3 and given by $T(x_1, x_2)$, then together with the classical Fourier's law of heat conduction the energy equation gives rise to the two-dimensional Laplace's equation

$$\frac{\partial^2 T}{\partial x_k \partial x_k} = 0 \text{ in } R^{\pm}. \tag{1}$$

Note that the Einsteinian convention of summing over a repeated index is adopted for latin subscripts running from 1 to 2.

The bond between the materials in R^+ and R^- at the interface Γ is microscopically damaged. The microscopic damage is assumed to be uniformly distributed over the interface. A macroscopic model for the heat conduction across the imperfect interface is given by

$$k^+ \left.\frac{\partial T}{\partial x_2}\right|_{x_2=0^+} = k^- \left.\frac{\partial T}{\partial x_2}\right|_{x_2=0^-} = \lambda \Delta T(x_1) \quad \text{for } x_1 \in [a, b], \tag{2}$$

where k^+ and k^- are the (constant) thermal conductivities of the materials in R^+ and R^- respectively, λ is a given positive coefficient, and $\Delta T(x_1) = T(x_1, 0^+) - T(x_1, 0^-)$ is the temperature jump across the interface. If the microscopic damage is uniformly distributed over the interface then λ is a constant.

At each and every point on the exterior boundary $C = C^+ \cup C^-$, either the temperature T or its normal flux $-k(x_1, x_2) n_p \partial T / \partial x_p$ (but not both) is specified. (Note that $k(x_1, x_2)$ denotes the thermal conductivity at the point (x_1, x_2) in the bimaterial and $[n_1(x_1, x_2), n_2(x_1, x_2)]$ is the unit normal vector to C at (x_1, x_2) which points out of the region R enclosed by C.)

The problem is to determine the temperature in the body by solving (1) subject to the boundary condition on C and the interface condition as given by (2).

Hypersingular boundary integral formulation

For $\xi_2 \neq 0$, guided by the analysis in Clements [10], one may derive a boundary integral solution for (1) in the form

$$\begin{aligned}\gamma(\xi_1,\xi_2)T(\xi_1,\xi_2) = \oint_C k(x_1,x_2)[T(x_1,x_2)n_p(x_1,x_2)\frac{\partial}{\partial x_p}\Phi(x_1,x_2,\xi_1,\xi_2) \\ - \Phi(x_1,x_2,\xi_1,\xi_2)n_p(x_1,x_2)\frac{\partial}{\partial x_p}T(x_1,x_2)]ds(x_1,x_2) \\ - k^+\int_a^b \Delta T(x_1)\left.\frac{\partial}{\partial x_2}\Phi(x_1,x_2,\xi_1,\xi_2)\right|_{x_2=0^+} dx_1, \qquad (3)\end{aligned}$$

where $\gamma(\xi_1,\xi_2) = 1$ if (ξ_1,ξ_2) lies inside R^+ or R^-, $0 < \gamma(\xi_1,\xi_2) < 1$ if (ξ_1,ξ_2) lies on C^+ or C^- [$\gamma(\xi_1,\xi_2) = 1/2$ if (ξ_1,ξ_2) lies on a smooth part of C^+ or C^-] and

$$\begin{aligned}\Phi(x_1,x_2,\xi_1,\xi_2) &= \frac{1}{2\pi}[\frac{1}{k^+}H(x_2) + \frac{1}{k^-}H(-x_2)]\,\mathrm{Re}\{\ln(z-c)\} \\ &\quad +\Psi(x_1,x_2,\xi_1,\xi_2), \qquad (4)\end{aligned}$$

with $z = x_1+ix_2$, $c = \xi_1+i\xi_2$, $i = \sqrt{-1}$, $H(x)$ being the Heaviside unit-step function and $\Psi(x_1,x_2,\xi_1,\xi_2)$ being given by

$$\begin{aligned}\Psi(x_1,x_2,\xi_1,\xi_2) &= -\frac{\mu}{k^+}H(x_2)\,\mathrm{Re}\{H(-\xi_2)\ln(z-c) + H(\xi_2)\ln(z-\overline{c})\} \\ &\quad +\frac{\mu}{k^-}H(-x_2)\,\mathrm{Re}\{H(-\xi_2)\ln(\overline{z}-c) \\ &\quad +H(\xi_2)\ln(\overline{z}-\overline{c})\}. \qquad (5)\end{aligned}$$

where $\mu = (k^- - k^+)/[2\pi(k^- + k^+)]$ and the bar denotes the complex conjugate of a complex number.

Note that $\Phi(x_1,x_2,\xi_1,\xi_2)$ as given by (4) together with (5) is the Green's function for the perfect interface, i.e. it satisfies the conditions

$$\Phi(x_1,0^+,\xi_1,\xi_2) - \Phi(x_1,0^-,\xi_1,\xi_2) = 0 \text{ for } -\infty < x_1 < \infty, \qquad (6)$$

$$\begin{aligned}k^+\left.\frac{\partial}{\partial x_2}\Phi(x_1,x_2,\xi_1,\xi_2)\right|_{x_2=0^+} &= k^-\left.\frac{\partial}{\partial x_2}\Phi(x_1,x_2,\xi_1,\xi_2)\right|_{x_2=0^-} \\ \text{for } -\infty &< x_1 < \infty. \qquad (7)\end{aligned}$$

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Differentiating (3) partially with respect to ξ_2 and then letting (ξ_1, ξ_2) approach a point on the imperfect interface, we find that the interfacial condition (2) may be re-written as

$$k^+ \oint_C k(x_1,x_2)[T(x_1,x_2)n_p(x_1,x_2)\ [\frac{\partial^2}{\partial x_p \partial \xi_2}\Phi(x_1,x_2,\xi_1,\xi_2)]\Big|_{\xi_2=0^+}$$
$$-n_p(x_1,x_2)\frac{\partial}{\partial x_p}T(x_1,x_2)\ [\frac{\partial}{\partial \xi_2}\Phi(x_1,x_2,\xi_1,\xi_2)]\Big|_{\xi_2=0^+}]ds(x_1,x_2)$$
$$+\frac{k^+k^-}{\pi(k^+ + k^-)}\mathcal{H}\int_a^b \frac{\Delta T(x_1)}{(\xi_1 - x_1)^2}dx_1 \ = \lambda \Delta T(\xi_1) \text{ for } a < \xi_1 < b, \qquad (8)$$

where $\mathcal{H}$ denotes the integral over the interval $[a, b]$ is to be interpreted in the Hadamard finite-part sense, i.e.

$$\mathcal{H}\int_a^b \frac{\Delta T(x)dx}{(\xi - x)^2} \stackrel{\text{def}}{=} \lim_{\sigma \to 0^+}[\int_a^b \frac{(\xi - x)^2 \Delta T(x)dx}{[(\xi - x)^2 + \sigma^2]^2} - \frac{\pi}{2\sigma}\Delta T(\xi)] \text{ for } a < \xi < b. \qquad (9)$$

A simple numerical procedure

For (3) to give an explicit expression for the required temperature field, the unknown quantities on $C \cup \Gamma$, i.e. T and $kn_p\partial T/\partial x_p$ on C and ΔT on Γ, must be determined. A simple numerical procedure for finding these unknowns from (3) and (8) is as follows.

The boundary C is discretized into M straight line elements denoted by $C^{(1)}$, $C^{(2)}$, $\cdots$, $C^{(M-1)}$ and $C^{(M)}$. Over the element $C^{(m)}$, T and $kn_p\partial T/\partial x_p$ are approximated as constants, i.e.

$$T(x_1,x_2) \simeq T^{(m)} \text{ and } k(x_1,x_2)n_p(x_1,x_2)\frac{\partial T}{\partial x_p} \simeq H^{(m)} \text{ for } (x_1,x_2) \in C^{(m)}, \qquad (10)$$

where $T^{(m)}$ and $H^{(m)}$ are constants.

From the condition given on the exterior boundary C, either T or $kn_p\partial T/\partial x_p$ is known on a boundary element. Thus, there are M unknown constants in (10) to be determined.

The interval $[a, b]$ representing the interface Γ is divided into L subintervals denoted by $[x^{(0)}, x^{(1)}]$, $[x^{(1)}, x^{(2)}]$, $\cdots$, $[x^{(L-2)}, x^{(L-1)}]$ and $[x^{(L-1)}, x^{(L)}]$, with $x^{(0)} = a$ and $x^{(L)} = b$. The interfacial temperature jump ΔT is approximated using

$$\Delta T(x_1) \simeq J^{(\ell)} \text{ for } x_1 \in [x^{(\ell-1)}, x^{(\ell)}], \qquad (11)$$

where $J^{(\ell)}$ are unknown constants to be determined.

With (10) and (11), if we let (ξ_1, ξ_2) in (3) be given by the midpoints of $C^{(r)}$ for $r = 1, 2, \cdots, M$, and if we take ξ_1 in (8) to be given by $(x^{(j-1)} + x^{(j)})/2$ for $j = 1, 2, \cdots, L$, we obtain a system of $M + L$ linear algebraic equations in $M + L$ unknowns. Once these unknowns are determined, (3) can be used to compute T approximately at any interior point in the bimaterial.

A specific example

To test the proposed numerical procedure, let us take R^+ to be the region $0 < x_1 < 1$, $0 < x_2 < 1/2$, with $k^+ = 1/5$, and R^- to be $0 < x_1 < 1$, $-1/2 < x_2 < 0$, with $k^- = 1/2$. On the interface between R^+ and R^-, i.e $0 < x_1 < 1$, $x_2 = 0$, we impose the condition (2) with $\lambda = 1$.

A solution of (1) which satisfies the interface condition (2) with $k^+ = 1/5$, $k^- = 1/2$ and $\lambda = 1$ is given by

$$\begin{aligned} T(x_1, x_2) &= \{H(x_2)[2\cos(x_2) + 5\sin(x_2)] \\ &\quad + H(-x_2)[\cos(x_2) + 2\sin(x_2)]\}\exp(-x_1) \end{aligned} \tag{12}$$

To devise a test problem, let us use (12) to generate boundary values of the temperature T on the sides $x_2 = \pm 1/2$, $0 < x_1 < 1$, and boundary values of the normal derivative of T on $x_1 = 0$, $-1/2 < x_2 < 1/2$ and also on $x_1 = 1$, $-1/2 < x_2 < 1/2$. The proposed numerical procedure is then applied to solve (1) subject to the boundary data thus generated and the interface condition (2). If it really works, we should be able to recover the solution (12) and the corresponding interfacial temperature jump $\Delta T(x_1) = \exp(-x_1)$ approximately.

For the practical implementation of the numerical procedure, each of the 4 sides of the bimaterial is divided into N equal length boundary elements (so that $M = 4N$) and the interface $[0, 1]$ into L equal subintervals. To avoid ambiguity, we require a boundary element to be in either R^+ or R^- but not partly in both the regions. One of the endpoints of the element is allowed to be on Γ, however. Thus, N must be chosen to be an even integer.

In Table 1, we compare the numerical values of the temperature T at selected points in the interior of the bimaterial, as computed using (3) with $(M, L) = (40, 5)$ and $(M, L) = (120, 15)$, with the exact solution (12). The two sets of numerical values show good agreement with the exact solution, except at points that are very close to the exterior boundary or the

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interface, e.g. $(0.5000, 0.4950)$ and $(0.7500, 0.0050)$, where there is a relatively higher percentage of error. The adverse effect of the boundary on the accuracy of the numerical values at points that are at a distance much smaller than the lengths of nearby elements is a well known phenomenon in boundary element research. One way of improving the numerical values at those points is to refine the discretization of the nearby boundary. This is clearly shown in Table 1 by the fact that the numerical values of the temperature at $(0.5000, 0.4950)$ and $(0.7500, 0.0050)$ improve significantly in accuracy when we treble the number of subintervals on the interface and also the boundary elements. The percentage errors of T at $(0.5000, 0.4950)$ and $(0.7500, 0.0050)$ are reduced further to about 0.0002% and 0.15% respectively when we use $(M, L) = (240, 30)$ in our computation.

Table 1. A comparison of the numerical and exact values of T with the exact ones at various points in the interior of the bimaterial.

(x_1, x_2)	$(M, L) = (40, 5)$	$(M, L) = (120, 15)$	Exact
$(0.8000, 0.3000)$	1.5254	1.5229	1.5225
$(0.7000, -0.2000)$	0.2883	0.2892	0.2894
$(0.1000, 0.4000)$	3.4261	3.4283	3.4286
$(0.3000, -0.1000)$	0.5897	0.5895	0.5892
$(0.5000, 0.4950)$	2.2966	2.5074	2.5081
$(0.7500, 0.0050)$	1.0132	0.9713	0.9565

Table 2. A comparison of the numerical and exact values of the interfacial temperature jump ΔT at various points on the interface.

(x_1, x_2)	$(M, L) = (40, 5)$	$(M, L) = (120, 15)$	Exact
$(0.1000, 0)$	0.8918	0.9006	0.9048
$(0.3000, 0)$	0.7376	0.7400	0.7408
$(0.5000, 0)$	0.6060	0.6064	0.6065
$(0.7000, 0)$	0.4980	0.4970	0.4966
$(0.9000, 0)$	0.4130	0.4085	0.4066

In Table 2, the numerical values of the interfacial temperature jump ΔT, as obtained using $(M, L) = (40, 5)$ and $(M, L) = (120, 15)$, are compared with the exact values at selected points on the interface. There is a

definite improvement in the numerical values when the discretization of the imperfect interface and the exterior boundary of the bimaterial is refined.

Conclusion

A hypersingular boundary integral formulation is derived for the plane steady-state heat conduction in a bimaterial with a microscopically imperfect interface. It is used to devise a boundary element method for computing approximately the temperature distribution in the bimaterial. The numerical result obtained for a specific test problem indicates that the method is capable of delivering accurate approximate solution for the problem under consideration.

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DOMAIN INTEGRAL EVALUATION FOR THERMOELASTIC STRESS ANALYSIS IN PRESSURE DIE CASTING

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SUMMARY

In this paper an efficient three-dimensional elastostatic stress model for the pressure die casting process is described. The collocation based boundary element method is used for the prediction of transient stress fields over a thermally stabilised casting cycle. The pressure die casting process suffers transient thermal penetration into the die over regions close to the surface of the die cavity and nozzle. Transient thermal fields give rise to domain integrals in the boundary element stress formulation. Two methods for evaluating these integrals are presented in the paper: (i) a simplex method on a mesh local to the cavity surface; (ii) a modified reciprocity method utilising Gaussian radial-basis functions, interpolating on a perturbed thermal field. The modified reciprocity method involves the evaluation of integrals incorporating two radial distances. The importance of high accuracy in the evaluation of these integrals is discussed. Also highlighted is the importance of the correct location of radial-function collocation points.The simplex method involves the application of a recursive scheme enabling integrals on tetrahedral elements to be reduced to line integrals. This approach is shown to be superior, providing high accuracy, stability and computational efficiency. Numerical experiments are performed to demonstrate the computational effectiveness of the approach.

KEY WORDS: casting; boundary elements; elastostatics, domain integrals.

1. INTRODUCTION

Pressure die casting is an important industrial process used for the mass production of complex components. A cyclic boundary element stress model has been developed so that stress levels can be predicted and examined. The model is based on the 3-D thermoelastic boundary element method and produces stresses and deformation due to both mechanical and thermal loads on the system. The thermal loading generates domain integrals in the thermoelastic BEM involving the transient temperature field, which require evaluation**.** Two methods investigated to evaluate the domain integral are domain meshing in the transient region and a form

of the dual reciprocity method. With the second approach domain meshing is avoided, however collocation points within the domain are required at additional computational cost. A highly accurate semi-analytical scheme is employed for the numerical evaluation of the integrals generated by this method. The first approach involves meshing the domain with tetrahedral elements and then applying recursive radial integration to these tetrahedral elements. Repetitive application of the recursive scheme effectively reduces the initial volume integral to line integrals. Excellent accuracy is obtained with the scheme. The two schemes have been compared in a set of numerical tests.

2. THERMOELASTIC BOUNDARY ELEMENT MODEL

In this section a three-dimensional thermoelastic BE model used for predicting die deformation and stress levels in the high pressure die casting process is described. This boundary element model utilises the predicted thermal behaviour of the die blocks and component together with the applied mechanical loads within the casting process. The boundary conditions on the various surfaces of a die block are as follows: *Cavity surface* Γ^{C}; Traction, $p = -P_c n$, where P_c is the injection pressure. *Die interface* Γ^{SS}; Traction, $p = -P_D n$, where P_D is the interface pressure. *Cooling channel surface* Γ^{W}; pressure $p \approx 0$. *External surface* Γ^{E}; $p = 0$. *Surface connected to sliders* Γ^{S}; $u_k = 0$, where k depends on the constraints of the slider.
A fully discretised form for the thermoelastic BEM formulation can be written

$$c_{\ell k}(x)u_k(x) + \sum_{n=1}^{N^k}\sum_{m=1}^{M^n} u_k^m \int_{\Delta_n^k} p_{\ell k}^*(x,y)\Theta^m \Gamma(y) - \sum_{n=1}^{N^k}\sum_{m=1}^{M^n} p_k^m \int_{\Delta_n^k} u_{\ell k}^*(x,y)\Theta^m d\Gamma(y) = \\ + \gamma \sum_{n=1}^{N^k}\sum_{m=1}^{M^n} T^m \int_{\Delta_n^k} u_{\ell k}^*(x,y)\Theta^m n_k d\Gamma(y) - \gamma \int_{\Omega^k} u_{\ell k}^*(x,y) T(y) d\Omega(z) \tag{1}$$

where all variables take their usual meaning, N^k is the number of elements, M^n is the number of nodes on the element, Θ^m is the shape function corresponding to node m and where the superscript m indicates the nodal values for displacement, traction, temperature and heat flux. The main problem with numerical implementation is the evaluation of the domain integral appearing on the right hand side of equation (1) involving the transient temperature field, which typically decays rapidly from the die cavity surface into the die, extending just 3-4 mm inwards.

3. MODIFIED RECIPROCITY METHOD

Domain meshing can be avoided by use of a modified form of the Dual Reciprocity Method. The modified DRM [1] is a suitable method as exponential type Gaussian

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Radial Basis Functions can be used to approximate the temperature field in the die and these are decay-type functions which can accurately represent the decay-type behaviour of the transient temperature field in this problem. The application of a modified reciprocity method to thermoelastic BEM results in singular surface integrals, the numerical evaluation of which can incur large errors. A semi-analytical integration scheme, based on Taylor expansions, is used to evaluate these integrals. The integrals can be written in the form $\int_\Gamma h(\underline{x})f(r)g(R)d\Gamma$, where the function f is singular, r and R are distances measured from a source point and a basis collocation point, respectively. The function $g(R)$ is replaced by $[g(R)-p(r)]+p(r)$ in the integral to produce $\int_\Gamma h(\underline{x})f(r)[g(R)-p(r)]d\Gamma + \int_\Gamma h(\underline{x})f(r)p(r)d\Gamma$. The polynomial $p(r)$ is selected to annihilate, where possible, the singularity in $\int_\Gamma h(\underline{x})f(r)[g(R)-p(r)]d\Gamma$, which consequently can be evaluated using standard quadrature techniques. The remaining integral $\int_\Gamma h(\underline{x})f(r)p(r)d\Gamma$ is of a simpler form and can be transformed into a contour integral, and evaluated numerically using Gauss-Legendre quadrature.

Consider the modified reciprocity method applied to a thermoelastic boundary element problem. A boundary element representation of this problem generates a domain integral of the form $\gamma\int_\Omega u^*_{\ell k}T_{,k}d\Omega$. In this case, $\nabla^2 u_{\ell k,k}=0$, making ∇^2 a suitable candidate for the adjoint operator. Consider then the existence of a function φ such that $\nabla^2\varphi = T+\varepsilon$. Substitution of this into the domain integral on the right-hand side of equation (1) gives

$$\int_\Omega u^*_{\ell k,k}Td\Omega = \int_\Omega u^*_{\ell k,k}\left(\nabla^2\varphi-\varepsilon\right)d\Omega =$$
$$\int_\Gamma\left(u^*_{\ell k,k}\frac{\partial\varphi}{\partial n}-\varphi\frac{\partial u^*_{\ell k,k}}{\partial n}\right)d\Gamma + \int_\Omega \nabla^2 u^*_{\ell k,k}\varphi d\Omega - \int_\Omega u^*_{\ell k,k}\varepsilon d\Omega = \int_\Gamma\left(u^*_{\ell k,k}\frac{\partial\varphi}{\partial n}-\varphi\frac{\partial u^*_{\ell k,k}}{\partial n}\right)d\Gamma \quad (2)$$

on discarding the domain integral $\int_\Omega u^*_{\ell k,k}\varepsilon d\Omega$. Consider φ represented by $\varphi=\sum_{j=1}^{M}a_j\gamma\circ R_j$, where $\nabla^2\gamma\circ R=\exp\left(-R^2/k\right)$, and its differentiable solution of the form $\gamma\circ R = -k^{3/2}\pi\,\mathrm{erf}\left(R/\sqrt{k}\right)/4R$, where Gaussian radial basis functions are selected, and where k is a shape parameter. Substituting for φ and $\partial\varphi/\partial n$ in equation (2) yields integrals of the form

$$\int_\Gamma\left(\frac{x_\ell}{r^3}\left(-\frac{k\exp(R^2/k)}{2R^2}+\frac{k^{3/2}\sqrt{\pi}\mathrm{erf}(R/\sqrt{k})}{4R^3}\right)\underline{R}\cdot\underline{n}-\left(\frac{-k^{3/2}\pi\mathrm{erf}(R/\sqrt{k})}{4R}\right)\left(\frac{n_\ell}{r^3}-\frac{3\,\underline{r}\cdot\underline{n}\,x_\ell}{r^5}\right)\right)d\Gamma \quad (3)$$

where n_ℓ and x_ℓ are components of $\underline{n}$ and $\underline{x}$, respectively. It is clear that the integrals in (3) are of the form $\int_\Gamma h(\underline{x})f(r)g(R)d\Gamma$ and can be transformed into a contour integral [1], and evaluated numerically using Gauss-Legendre quadrature.

4. RECURSIVE METHOD

In the case of application of the thermoelastic BEM to the die casting process and meshing the domain, results in the need to evaluate singular integrals over the discretised volume. The proposed method involves recursive radial integration on simplexes and is used to transform the singular domain integral over a volume element to a non-singular integral over a line, which can be evaluated to high accuracy using standard numerical quadrature. The domain integral present in the thermoelastostatic BEM can be represented in the following form, $\int_\Omega u^*_{\ell k,k} T dV$, where T is temperature, $u^*_{\ell k,k} = C\left(x_\ell - x^I_\ell\right)/\rho^3$, where C is related to material constants. Let Ω be a domain of dimension 3 and consider discretisation with 3-simplexes (tetrahedral elements). Over a 3-simplex S_3 a temperature field is approximated by a linear polynomial of the form $\left(a^j_0 + b^j_m x_m\right)T_j$, where summation over j is from 1 to 4 and m is from 1 to 3, with a^j_0 and b^j_m being constants. A two-stage recursion process reduces this volume integral to a line integral [2]. The first stage of recursive integration involves

$$\int_{S^k_2 - \Gamma^k_{\varepsilon_2}} \frac{1}{r_3^3} \int^{r_3} \frac{x_\ell - x^I_\ell}{\rho_3^3}\left(a^j_0 + b^j_m x_m\right)\rho_3^2 d\rho_3 d\Gamma_2 =$$
$$\left(\left(a^j_0 + b^j_m x^I_m\right)\int_{S^k_2 - \Gamma^k_{\varepsilon_2}} \frac{x_\ell - x^I_\ell}{r_3^3} d\Gamma_2 + \frac{b^j_m}{2}\int_{S^k_2 - \Gamma^k_{\varepsilon_2}} \frac{\left(x_\ell - x^I_\ell\right)\left(x_m - x^I_m\right)}{r_3^3} d\Gamma_2\right) \tag{4}$$

where k is summed from 1 to 4, and where S_2 is 2-simplex. Consider a second stage of recursion applied to the integral $\int_{S_2 - \Gamma_{\varepsilon_2}} \left(\left(x_\ell - x^I_\ell\right)/r_3^3\right) d\Gamma_2$. The results of the second-stage of recursion are

$$\int_{S_2 - \Gamma^k_{\varepsilon_2}} \frac{x_\ell - x^I_\ell}{r_3^3} d\Gamma_2 =$$
$$\underline{r}^k_2 \cdot \underline{n}^k_2 \left(-\int_{S^k_1 - \Gamma^k_{\varepsilon_1}} \frac{x_\ell - x^{II}_\ell}{r_2 r_3} d\Gamma_1 + \int_{S^k_1 - \Gamma^k_{\varepsilon_1}} \frac{x_\ell - x^{II}_\ell}{r_2^2} \ln\left(r_2 + r_3\right) d\Gamma_1 + \int_{S^k_1 - \Gamma^k_{\varepsilon_1}} \frac{x^I_\ell + x^{II}_\ell}{c^2_{II} r_2 r_3} d\Gamma_1\right) \tag{5}$$

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where k is summed from 1 to 3 and $c_{II} = \left\| \underline{x}^{II} - \underline{x}^{I} \right\|_2$. Inspection of equation (5) reveals that the remaining integrals are line integrals; these can be evaluated to high accuracy numerically [2] using standard quadrature techniques.

5. THERMOELASTOSTATIC PROBLEM

The accuracy of the recursive integration scheme and the modified reciprocity method are compared by considering a relatively simple elastostatic problem, consisting of an elastic body (a cube) subject to a uniform temperature increase. An analytical solution exists for this problem [3]. The dimensions, boundary conditions and thermo-mechanical properties of the cube are illustrated in Figure 1(a). In the case of the simplex method, solid elements (tetrahedrals) were created first and subsequently the nodes and faces of these tetrahedral elements that lay on the surface of the cube were used as the surface mesh of linear triangular elements for the boundary element analysis. The mesh utilised consisted of 8 nodes and 6 tetrahedral elements; this mesh is shown in Figure 1(b). The recursive scheme employed 2 levels of recursion, with 8 point Gauss-Legendre quadrature being used on each 1-simplex. For the application of the modified reciprocity (ModR) method, two different meshes were tested and these are depicted in Figure 2. Mesh 1 and 2 consist of 24 elements, 14 nodes and 396 elements, 200 nodes respectively. A total of 22 basis collocation points were used in conjunction with both meshes. The positions of the 22 basis-collocation points are: 14 are coincident with the nodes of the coarse surface mesh and 8 form a symmetric grid inside the cube. A value of 0.125 was used for the shape parameter k. The ModR method tested utilised semi-analytical integration with 256 points Gauss quadrature being used for the area integral and 4 point Gauss-Legendre quadrature for the line integral. The deformation obtained, using both methods, at internal points in the cube given by (d, d, d) with $50 \le d \le 100\text{mm}$, was compared against the analytical solution. The results of the tests are presented in Table 1 where it can be seen that for the modified reciprocity method reasonable accuracy is obtained, however, the error increases as the source point approaches the end face. Note, mesh density has little effect on error size, in addition, integration errors outweigh those present due to interpolation [1]. In contrast to the first method the recursive scheme delivers excellent accuracy independent of the position of the source point; another benefit of this scheme is that it does not require a shape parameter to be set.

CONCLUSIONS

Two methods for evaluating domain integrals present in thermoelastic BEM analysis are described in this paper, i.e. the Modified Reciprocity method and the Simplex recursion method. The Simplex recursion method required domain elements but produced superior accuracy and stability.

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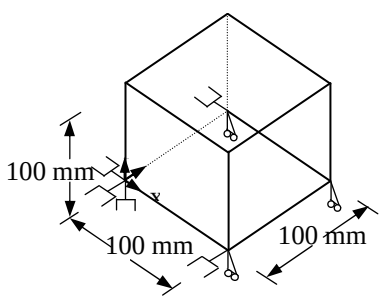
Distance (mm)	|error(%)|		
	ModR Method		Simplex Method
	Mesh 1	Mesh 2	Mesh 1
50	2.53	2.01	0.00
90	3.02	3.22	0.00
99	3.53	3.63	0.00
100	0.01	0.015	0.00

Table 1. Error in numerical predictions for a cube with uniform temperature load.



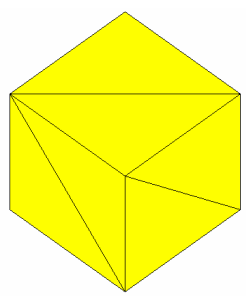
a) boundary conditions

$E=180\ kN/mm^2$

$\Delta T = 100^0\,C$

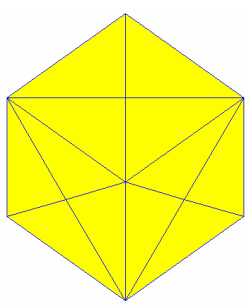
$\alpha = 10^{-5}\ {}^0C^{-1}$

$\nu = 0.3$



b) mesh used in simplex method

Figure 1. Boundary conditions and mesh utilised in simplex method.



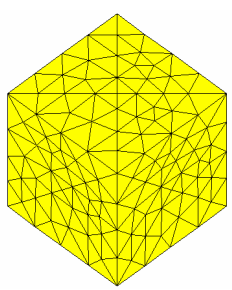


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Figure 2. Meshes used in modified DRM test.

The Cauchy Problem for the Steady-State Convection-Diffusion Equation using BEM

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Keywords: Cauchy problem; Inverse problem; Convection-Diffusion Equation; Helmholtz Equation; Boundary Element Method (BEM); Tikhonov Regularisation Method; Discrepancy Principle.

Abstract. In this paper, a numerical technique based on the BEM combined with the Tikhonov regularisation method is developed in order to solve the Cauchy problem associated to the steady-state convection-diffusion equation. The governing convection-diffusion equation is transformed into a Helmholtz or modified Helmholtz equation. The choice of the regularisation parameter is based on the discrepancy principle method.

Introduction

All over the world rivers and lakes are contaminated by several types of pollution, resulting in high health risk to people, animals and plants that are exposed to this water. The most important issue when trying to understand, and therefore to control, the pollution transport process is to know the origin of the source of the contamination. The aim of this mathematical study is to develop the necessary techniques to solve the practical problem of identifying this source of pollution when having a limited amount of measurement data taken from the water.

Mathematical formulation

Let us consider a bounded domain $\Omega \subset \mathbb{R}^d$, where d is the dimension of the space in which the problem is posed, and we assume that Ω is bounded by a surface Γ which may consist of several segments, each being sufficiently smooth in the sense of Liapunov. We also assume that the boundary of the domain, $\partial\Omega = \Gamma$, consists of three parts, Γ_0, Γ_1 and Γ_2, such that $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$, where $\Gamma_0, \Gamma_1 \neq \varnothing$, and that the three parts of the boundary do not intersect each other and each common boundary is smooth.

The water pollution problem can be modelled by the following steady-state convection-diffusion equation:

$$\sum_{i=1}^{d} \frac{\partial^2 c}{\partial x_i^2}(\underline{x}) - \sum_{i=1}^{d} u_i \frac{\partial c}{\partial x_i}(\underline{x}) - hc(\underline{x}) = 0, \qquad \underline{x} \in \Omega, \tag{1}$$

where $c(\underline{x})$ is the concentration of the pollutant, $u_i, i = 1, 2, ..., d$ are functions characterizing the flow velocity and h is the decay coefficient. In this study we assume that $u_i, i = 1, 2, ..., d$ and h are constants so we can apply the BEM.

Based on the change of variable $c = v \exp\left(\frac{1}{2}\sum_{i=1}^{d} u_i x_i\right)$ eq (1) may be recasted as follows:

$$\sum_{i=1}^{d} \frac{\partial^2 v}{\partial x_i^2} - \left(h + \frac{1}{4}\sum_{i=1}^{d} u_i^2\right) v = 0. \tag{2}$$

Thus we have reduced our problem to solving the Helmholtz, or the modified Helmholtz equation, depending on the sign of $\left(h + \frac{1}{4}\sum_{i=1}^{d} u_i^2\right)$. Associated to this equation, we consider the following boundary conditions:

$$v(\underline{x}) = f_0(\underline{x}), \qquad \underline{x} \in \Gamma_1 \tag{3}$$

$$\frac{\partial v}{\partial n}(\underline{x}) = f_1(\underline{x}), \qquad \underline{x} \in \Gamma_1 \cup \Gamma_2 \tag{4}$$

where it can be seen that the boundary Γ_1 is overspecified by the prescription of both v and $\frac{\partial v}{\partial n}$, while the boundary Γ_0 is underspecified, since both $v|_{\Gamma_0}$ and $\frac{\partial v}{\partial n}|_{\Gamma_0}$ are unknown and have to be determined. This situation occurs very often in practical problems, when one cannot make any measurements on a part of the boundary, but can obtain some extra information from the remaining part. The problem given by eq (2) and the boundary conditions (3) and (4) is termed the *Cauchy problem.* Although the problem may have a unique solution, it is well known that this solution is unstable with respect to small perturbations in the data on Γ_1 or Γ_2 and thus the problem is ill-posed and we cannot use a direct approach to solve the resulting system of linear equations which arises from discretising eq (2). Supposing that the boundaries Γ_0, Γ_1 and Γ_2 are discretised into N_0, N_1 and N_2 boundary elements, respectively, such that $N_0 + N_1 + N_2 = N$, the problem reduces to solving an ill-conditioned system of $2N$ equations in $2N_0 + N_2$ unknowns which can be generically written as, see [1],

$$\mathbb{A}\underline{v} = \underline{b}, \tag{5}$$

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where $\mathbb{A}$ is a $2N \times (2N_0 + N_2)$ matrix which depends only on the geometry of the boundary Γ, $\underline{v}$ is a vector of order $2N_0 + N_2$ containing the unknown values of v and $\frac{\partial v}{\partial n}$ and $\underline{b}$ is a vector of order $2N$, computed using the boundary conditions on Γ_1 and Γ_2. We apply the Tikhonov regularisation method, see [2], in order to deal with this ill-conditioned system of linear algebraic equations. The Tikhonov regularised solution of the system of equations (5) is given by

$$\underline{v}_\lambda : T_\lambda(\underline{v}_\lambda) = \min_{\underline{v} \in \mathbb{R}^{2N_0+N_2}} T_\lambda(\underline{v}), \tag{6}$$

where T_λ represents the Tikhonov functional given by $T_\lambda(\underline{v}) = \| \mathbb{A}\underline{v} - \underline{b} \|_2^2 + \lambda^2 \| \mathbb{I}\underline{v} \|_2^2$, where the identity matrix $\mathbb{I} \in \mathbb{R}^{(2N_0+N_2)\times(2N_0+N_2)}$ induces the smoothing norm $\| \mathbb{I}\underline{v} \|_2$ and $\lambda \in \mathbb{R}$, $\lambda \geq 0$ is the regularisation parameter to be chosen. Solving $\nabla T_\lambda(\underline{v}) = 0$, we obtain that the Tikhonov regularised solution $\underline{v}_\lambda$ is given by the solution of the regularised equation $\left(\mathbb{A}^T\mathbb{A} + \lambda^2\mathbb{I}^T\mathbb{I}\right)\underline{v}_\lambda = \mathbb{A}^T\underline{b}$.

The choice of the regularisation parameter is achieved by employing the discrepancy principle method, see [3]. According to this principle the regularisation parameter should be such that $\| \mathbb{A}\underline{v}_\lambda - \underline{b} \|_2 \approx \delta$, where δ is an estimate of the level of noise present in the problem, i.e. $\delta = \| \underline{b} - \underline{b}^\varepsilon \|_2$, where $\underline{b}^\varepsilon$ is the perturbed value of the right hand side of the system of equations (5).

Numerical Results

In order to investigate the performance of the proposed numerical method, we solve the Cauchy problem for some test examples in two different solution domains, namely an annular and a rectangular domain.

Example 1. The solution domain is the annulus $\Omega = \{(x, y) | 0.5 < x^2 + y^2 < 1\}$ and the boundary of this domain Ω is $\Gamma = \Gamma_0 \cup \Gamma_1$, where $\Gamma_0 = \{(x, y) : x^2 + y^2 = 1\}$ and $\Gamma_1 = \{(x, y) | x^2 + y^2 = 0.5\}$ (we have taken $\Gamma_2 = \varnothing$). The annular domain can be used to model an island surrounded by the water of a lake. In this case it is assumed that the measurements of the pollutant concentration can only be made at some locations on the edge of the island and that the lake was polluted from somewhere on the outer edge of the lake. We consider that the process of pollution of the lake is governed by the following convection-diffusion equation: $\nabla^2 c(x, y) + \frac{\partial c}{\partial x}(x, y) + 2\frac{\partial c}{\partial y}(x, y) - \frac{5}{4}c(x, y) = 0$, where $(x, y) \in \Omega$. This equation corresponds to the two-dimensional modified Helmholtz

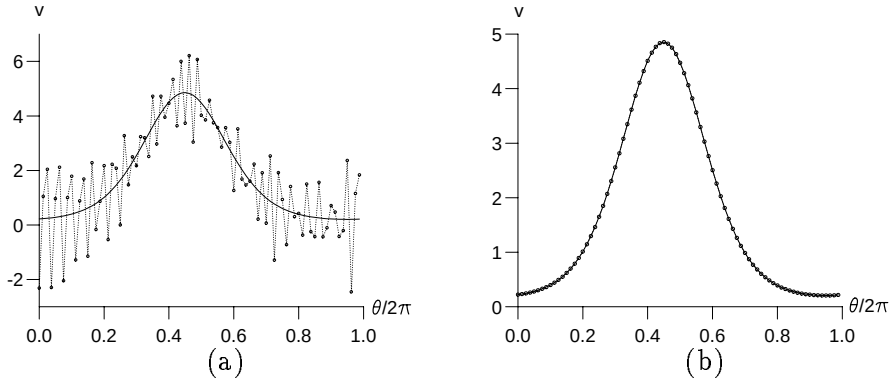

Figure 1: The numerical results obtained for v on the boundary Γ_0 using 160 $(\cdots\circ\cdots)$ boundary elements and the analytical solution (—), for the Cauchy problem considered in Example 1 with (a) $\lambda = 0$, and (b)$\lambda = 7 \cdot 10^{-11}$.

equation: $\nabla^2 v(x,y) - \frac{5}{2}v(x,y) = 0$, where $(x,y) \in \Omega$ and the analytical solution to be retrieved is given by $v(x,y) = \exp\left(-\frac{3}{2}x + \frac{1}{2}y\right)$.

We found that if 80 or less boundary elements are employed then the method provides a good numerical solution even if the system (5) is solved using Gauss elimination method without any regularisation. However, if a more accurate numerical solution is required then more boundary elements have to be employed. Fig. 1(a) shows the numerical results obtained when 160 boundary elements are used without regularisation. It can be seen that the numerical solution obtained using this direct approach contains large oscillations. This is due to the very small condition number cond($\mathbb{A}$)=det($\mathbb{A}\mathbb{A}^T$) of the matrix $\mathbb{A}$ of system of equations (5). Being too small, its exact value cannot be stored in the computer and this generates the errors in the numerical results. We can see now that the direct approach to this Cauchy problem produces a highly unstable solution and therefore we need to find a new approach. This is the reason why methods such as the Tikhonov regularisation method have to be employed. According to the discrepancy principle method, the optimal regularisation parameter is $\lambda = 7 \cdot 10^{-11}$. If we solve our problem using the Tikhonov regularisation method with the regularisation parameter chosen using the discrepancy principle, then we obtain the numerical results illustrated in Fig. 1(b) and it can be seen that the numerical solution is stable. In fact the amount of error (relative error) is less than $10^{-1}\%$ for the flux $\frac{\partial v}{\partial n}$ and less than $10^{-5}\%$

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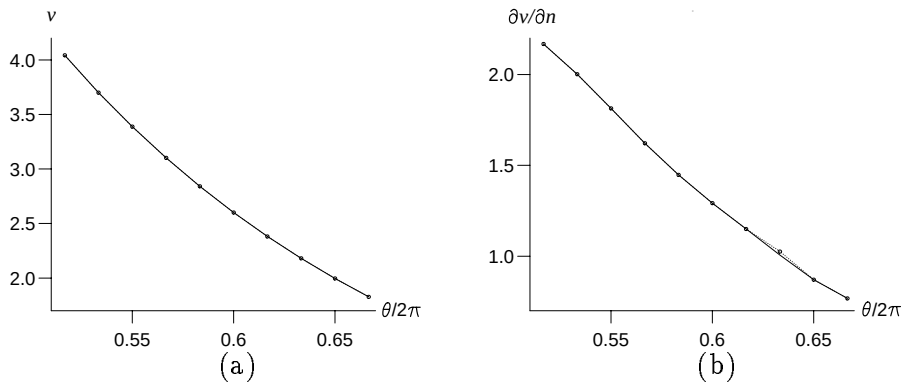

Figure 2: The numerical results obtained for (a) v, and (b) $\frac{\partial v}{\partial n}$, on the boundary Γ_0 using 60 ($\cdots\circ\cdots$) boundary elements and the analytical solution (—), for the Cauchy problem considered in Example 2 with $\lambda = 4 \cdot 10^{-16}$.

for v. When more boundary elements are employed, the accuracy of the numerical solution improves and thus we may conclude that our numerical method is convergent with respect to the mesh size discretisation.

Example 2. We consider a rectangular solution domain $\Omega = \{(x, y)| - 2 < x < 2, -1 < y < 1\}$ bounded by the rectangle $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$, where $\Gamma_0 = [(-2, 1), (-2, -1)]$, $\Gamma_1 = [(2, -1), (2, 1)]$ and $\Gamma_2 = [(2, 1), (-2, 1)] \cup [(-2, -1), (2, -1)]$. This geometry can be used to approximate the real geometry of a part of a river which is polluted somewhere upstream and where measurements of the concentration of the pollutant can only be made at some downstream locations in the river. The governing convection-diffusion equation is taken to be the following: $\nabla^2 c(x, y) - \frac{4}{3}\frac{\partial c}{\partial x}(x, y) = 0$, where $(x, y) \in \Omega$. This equation corresponds to the 2D Helmholtz equation: $\nabla^2 v(x, y) - \frac{4}{9} v(x, y) = 0$, where $(x, y) \in \Omega$ and the analytical solution to be retrieved is given by: $v(x, y) = \exp\left(-\frac{1}{2}x + \frac{\sqrt{7}}{6}y\right)$. We consider the same Cauchy problem, i.e. the values of the concentration and its normal derivative are known from the measurements taken from some downstream locations in the river and, by using these values and those of the normal derivative on the banks of the river, the method will provide the values of the concentration and its normal derivative at some upstream points in the river and, thus, the source of pollution can be located. As in the previous example, regularisation has to be applied in order to obtain a stable solu-

tion. When 60 boundary elements are employed, the optimal regularisation parameter found by using the discrepancy principle is $\lambda = 4{\cdot}10^{-16}$ and Figs. 2(a) and (b) illustrate the numerical solutions for v and $\frac{\partial v}{\partial n}$, respectively, on the unspecified boundary Γ_0, obtained when using the BEM. Again we observe that the numerical solution is stable. Furthermore, as the number of boundary elements employed by the BEM increases, then the numerical solution obtained becomes more accurate.

Conclusions

In this paper we have presented a numerical technique for solving an inverse problem associated to the steady-state convection-diffusion equation which is based on the BEM combined with the Tikhonov regularisation method. The technique reduces the convection-diffusion equation to the Helmholtz or modified Helmholtz equation and then applies the Tikhonov regularisation method to the ill-conditioned system of linear algebraic equations obtained by an application of the BEM. The discrepancy principle method is employed when choosing the optimal regularisation parameter. Numerical results have been obtained for numerous problems, but only two examples have been presented in this paper. All the results obtained, for various numbers of boundary elements, have shown that the technique solves the problem accurately, as the numerical results were found to be stable and convergent, for both an annular and a rectangular domain. Future work consists in testing the method proposed for random noise added into the input data f_0 and f_1.

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