Section Five

Sensitivity Analysis and Optimisation

Shape Sensitivity of the Anisotropic Elastic Response

Lucía Comino[1] and Rafael Gallego[2]

[1] Dept. Structural Mechanics, University of Granada, Politécnico de Fuentenueva, 18071 Granada, Spain, lcomino@ugr.es
[2] Dept. Structural Mechanics, University of Granada, Politécnico de Fuentenueva, 18071 Granada, Spain, gallego@ugr.es

Keywords: Shape Sensitivity, Two-dimensional Anisotropic Elasticity, Inverse Problem, Boundary Element Method (BEM)

Abstract. Flaw identification with non destructive experimental techniques can be modelled through the so-called inverse problem. This communication deals with problems where the unknowns are the location and shape of cavities embedded in a linear elastostatic domain. The unknowns are sought as to achieve the best fit between measured and computed values of some physical quantities (response of the system). This usually leads to the minimization of a cost functional. The most efficient non-linear minimization algorithms need the computation of the gradient of objective function under changes in the unknown parameters of the problem, in this case, the geometry of the model.

There are several approaches to calculate this shape sensitivity, here it is emphasized the effectiveness of the Adjoint Variable Method, which establishes a formula expressed in terms of boundary integrals, suitable for Boundary Element Method implementation.

Numerical results are presented for plane problems with generalized rectilinear anisotropy, in multi-connected homogeneous domain. They are compared with finite difference evaluations and the efficiency of the proposed Adjoint Variable strategy is proved.

Introduction

One of the most important cases treated with non-destructive evaluation techniques, is flaw detection in solids. This kind of problems can be solved as an *Inverse Problem*. In shape optimization, as well as in identification inverse problems, the geometrical domain plays a major role and the need to compute the sensitivity of integral functionals with respect to shape parameters, arises in both situations.

This communication is focused in some geometrical inverse problem like cavity detection, where part of the domain boundary is unknown. Its determination is attempted by minimizing an integral functional, which expresses the difference between computed and measured data on a part of the external domain, e.g. in the form of least-squares distance. Most usual optimization algorithms uses first order derivatives, *sensitivities*, which evaluation could be based on numerical differentiation techniques. But apart from

being computationally expensive, it constitutes an ill-posed mathematical problem, then, the analytical differentiation is proposed here. A formulation based on the *Adjoint Variable MethodAV* is presented, and the corresponding boundary-only formula for the shape sensitivity. This approach seems to be the most efficient since just one adjoint problem has to be solved for each functional to be minimized [7]. Beside, since the domain is a primary unknown, it is suitable for using Boundary Integral Equation (BIE) techniques, which offer the minimal modeling.

BEM has been implemented for plane problems with generalized rectilinear anisotropy, in multi-connected homogeneous domain. The results has been compared with finite-difference evaluations.

Boundary Integral Equation. Anisotropic Fundamental Solution

The displacements of the direct problem satisfy the following BIE:

$$c_{ij}(\mathbf{y})u_j(\mathbf{y}) + \int_\Gamma T_{ij}(\mathbf{z},\mathbf{y})u_j(\mathbf{z})d\Gamma = \int_\Gamma U_{ij}(\mathbf{z},\mathbf{y})t_j(\mathbf{z})d\Gamma \tag{1}$$

being T_{ij} and U_{ij} the anisotropic displacement and stress fundamental solutions [1] [4].

$$U_{ij}(\mathbf{z},\mathbf{z}') = 2\Re\left[p_{j1}A_{i1}\ln(z_1 - z_1') + p_{j2}A_{i2}\ln(z_2 - z_2')\right] \tag{2}$$

$$T_{ij}(\mathbf{z},\mathbf{z}') = 2\Re\left[\frac{q_{j1}A_{i1}}{z_1 - z_1'}(\mu_1 n_1 - n_2) + \frac{q_{j2}A_{i2}}{z_2 - z_2'}(\mu_2 n_1 - n_2)\right] \tag{3}$$

Shape Sensitivity

Inverse Problem The present work deals with application of BIE to some identification inverse problems. The formulation of shape sensitivity may result form either the direct differentiation approach or the adjoint variable method. For this particular problem, where a singular functional is to be minimized, the *AV* approach is very attractive since the analytical gradient expression of the cost functional, is a boundary integral that involves values from the primary (forward problem) and the adjoint state.

Adjoint Variable Method The primary problem we consider, is a bounded domain Ω with an external boundary $\delta\Omega$, and internal cavities Ω^- with a traction-free boundary Γ. The governing equations of the problem, are the ones of linear elastostatic domain, with no body forces. The shape and position of the boundary Γ are the unknowns of the problem. The usual approach for finding Γ is the minimization of an integral functional $\mathfrak{J}$, being J the distance between $\mathbf{u}_\Gamma$ and/or $\mathbf{t}_\Gamma$ (computed displacements and tractions for a given location of Γ), and $\widehat{\mathbf{u}}$ and/or $\widehat{\mathbf{t}}$ (measured values in surfaces C_u, C_t, part of external boundary $\delta\Gamma$)

$$\mathfrak{J}(\Gamma) = J(\mathbf{u}_\Gamma, \mathbf{t}_\Gamma) = \int_{C_u} j_u(\mathbf{u},\widehat{\mathbf{u}})dS + \int_{C_t} j_t(\mathbf{t},\widehat{\mathbf{t}})dS \tag{4}$$

with $j_u = \frac{1}{2}|\mathbf{u} - \widehat{\mathbf{u}}|^2$ and $j_t = \frac{1}{2}|\mathbf{t} - \widehat{\mathbf{t}}|^2$ usual least-squares distance.

ISBN 0904 188965

Minimization of $\mathfrak{J}$ is attempted as the minimization of J subject to $\mathbf{u} = \mathbf{u}_\Gamma$ and $\mathbf{t} = \mathbf{t}_\Gamma$; *gradient minimization methods* are the most common for this purpose [3]. This implies computations of the derivative of $\mathfrak{J}$ with respect to (the flaw design parameters) Γ. To investigate the variations of $\mathfrak{J}$, the *material differentiation* notion has been introduced. The geometrical transformation is described as

$$\mathbf{x} \in \Gamma \Rightarrow \mathbf{x} + \delta\mathbf{x} = \mathbf{x} + \boldsymbol{\Theta}(\mathbf{x})\boldsymbol{\tau} \tag{5}$$

being $\boldsymbol{\Theta}$ the transformation velocity of the boundary Γ, and $\boldsymbol{\tau}$ a design time-like parameters vector [8]. We will note with $*(\mathbf{x}, \tau)$ the material derivative of a field produced by the geometrical transformation.

We start from an augmented funcional $\mathfrak{L}$ of J:

$$\mathfrak{L}(\mathbf{u}, \mathbf{w}; \Gamma) = J(\mathbf{u}, \mathbf{t}) + \mathfrak{A}(\mathbf{u}, \mathbf{w}; \Gamma) \tag{6}$$

being $\mathfrak{A}$ is the weak formulation of the forward problem,

$$\mathfrak{A}(\mathbf{u}, \mathbf{w}; \Gamma) = \int_\Omega [\boldsymbol{\sigma}(\mathbf{u}) : \nabla\mathbf{w}]dV + \int_{\delta\Omega} (\boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{n} \cdot \mathbf{w})dS = 0 \tag{7}$$

using the space of test functions $\mathfrak{V} = \{\mathbf{w} \in \{H^1_{loc}(\Omega)\}^3\}$, $\mathbf{w}$ acts as Lagrange multiplier. If the *material differentiation* is applied to $\mathfrak{L}$ [10], we obtain:

$$\overset{*}{\mathfrak{L}}(\mathbf{u}, \mathbf{w}; \Gamma) = \mathfrak{L}_{,\mathbf{u}} \cdot \mathbf{u}_{,\tau} + \mathfrak{L}_{,\Gamma} \cdot \Theta \tag{8}$$

$$\begin{aligned} \mathfrak{L}_{,\mathbf{u}} \cdot \mathbf{u}_{,\tau} &= \int_{C_u} (j_{u,u} \cdot \mathbf{u}_{,\tau})dS + \int_{C_t} (j_{t,t} \cdot \mathbf{t}_{,\tau})dS + \int_\Omega [\boldsymbol{\sigma}(\mathbf{u}_{,\tau}) : \nabla\mathbf{w}]dV \\ \mathfrak{L}_{,\Gamma} \cdot \Theta &= \int_\Omega ([\boldsymbol{\sigma}(\mathbf{u}) : \nabla\mathbf{w}]div\boldsymbol{\Theta})dV - \int_\Omega ([\boldsymbol{\sigma}(\mathbf{u}) \cdot \nabla\mathbf{w} + \boldsymbol{\sigma}(\mathbf{w}) \cdot \nabla\mathbf{u}] : \nabla\boldsymbol{\Theta})dV \end{aligned} \tag{9}$$

Actual variations of $\mathfrak{L}$ can be expected only when $\Theta \neq 0$, so the Lagrange multiplier $\mathbf{w}$ can be chosen from

$$\mathfrak{L}_{,\mathbf{u}}(\mathbf{u}_\Gamma, \mathbf{w}; \Gamma) \cdot \mathbf{u}_{,\tau} = 0 \quad \forall \mathbf{u}_{,\tau} \in \mathfrak{V} \tag{10}$$

This equation represents the weak formulation of an elastostatic well posed problem that will be called *adjoint state* and which strong formulation is given by

$$\begin{aligned} div(\boldsymbol{\sigma}(\mathbf{w})) &= 0 \quad en \quad \Omega \\ \mathbf{w} &= \bar{\mathbf{w}} = +j_{,t} \quad en \ C_t \\ \mathbf{w} &= 0 \quad en \ \delta\Omega_u/C_t \\ \mathbf{t} &= \bar{\mathbf{t}} = -j_{,u} \quad en \ C_u \\ \mathbf{t} &= 0 \quad en \ \delta\Omega_t/C_u \end{aligned} \tag{11}$$

Then, the material derivative of $\mathfrak{J}$ can be computed by:

$$\overset{*}{\mathfrak{J}} = \overset{*}{\mathfrak{L}} = \mathfrak{L}_{,\Gamma}(\mathbf{u}_\Gamma, \mathbf{w}_\Gamma; \Gamma) \cdot \boldsymbol{\Theta} \tag{12}$$

This equation can be manipulated to get a boundary expression. If the Divergence Theorem and the boundary conditions of the primary problem are applied, then we make use of the tangential gradients decomposition [6], and we use some local reference axis ($x_1' = \mathbf{n}, x_2' = \mathbf{t}$), a compact formula for the shape sensitivity is obtained:

$$\overset{*}{\mathfrak{J}} = \frac{d\mathfrak{J}}{d\tau} = \int_\Gamma [\sigma_{22}'(\mathbf{u}) : \mathbf{w}_{2,2}'] \mathbf{\Theta}_n dS \tag{13}$$

with $\mathbf{\Theta}_n$ the normal transformation velocity component.

Moreover, the primary and the adjoint problems are associated with the same integral governing operator and boundary conditions of the same type. Thus, the operator matrix to be built and factored, in order to apply BIE techniques is used for both problems.

Parametrization The variation of the geometry is always represented by a parametrization. Here, a concept developed by Gallego and Suárez [9] has been used. Consists in defining directly the modification field instead of the geometry. This means applying a deformation field to some initial geometry, which is able to move it until any possible solution. The parametrization is defined as follows:

$$\delta x_i(\mathbf{x}) = \mathbf{\Theta}_{ip} \delta \tau_p \tag{14}$$

$\mathbf{\Theta}_{ip}$ is the parametrization matrix and $\delta\tau_p$ is a vector with p parameters τ.

A linear deformation field in 2D has been chosen, then 6 parameters are needed. The parametrization matrix and the parameters vector are expressed in the following way [5] (with $\mathbf{x}_i = \mathbf{x}_i^{real} - \mathbf{x}_i^{cg}$):

$$\mathbf{\Theta}_{ip} = \begin{bmatrix} 1 & 0 & x_2 & x_1 & x_1 & x_2 \\ 0 & 1 & -x_1 & x_2 & -x_2 & x_1 \end{bmatrix} \tag{15}$$

$$\delta\tau_p = \begin{bmatrix} \delta x_1^{cg} \\ \delta x_2^{cg} \\ \delta\omega \\ \delta\epsilon_m \\ \delta\epsilon_l \\ \delta\epsilon_{12} \end{bmatrix} = \begin{bmatrix} \textit{first coordinate of the centroid of the flaw} \\ \textit{second coordinate of the centroid of the flaw} \\ \textit{rotation angle} \\ \textit{spherical strain} \\ \textit{horizontal elongation} \\ \textit{distortion} \end{bmatrix} \tag{16}$$

Numerical results

To prove the efficiency of the proposed *AV* approach , several numerical tests have been done, and the results have been compared with the sensitivities obtained with finite-difference. An orthotropic (birch plywood) square plane plate (side=12m) has been studied. Two cases have been considered, one with a square cavity (side=2m) and one with a circular void (diameter=2m). The discretization of the model has been made with an increasing number of isoparametric cuadratic elements. Fig.1 shows the cases studied. Measurements of u_2 have been made in the left-hand side of the square plate. The numerical results are shown in the following tables and graphs (Fig. 2).

ISBN 0904 188965

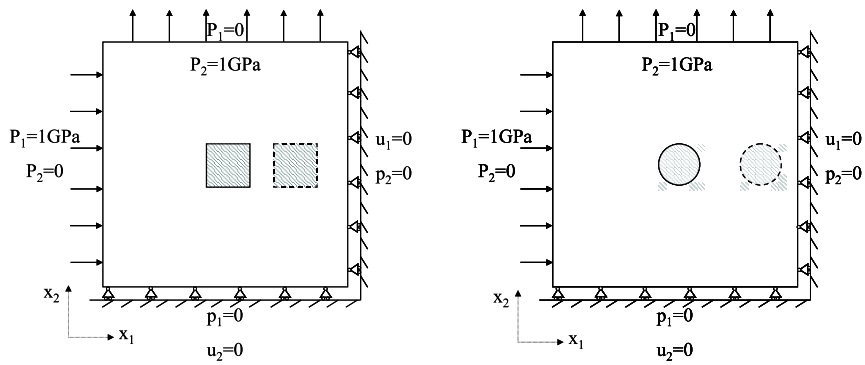


Figure 1: Orthotropic plates with internal cavities in theirs real and supposed positions

Summary and conclusion

In the present study, a shape sensitivity analysis for identification of internal defects as voids, in anisotropic 2D multi-connected domain, has been presented. The *adjoint variable* approach has been developed in connection with BIE formulations of the direct problem. A general formulation for the sensitivity of an objective functional is described and expressed as a compact boundary integral formula that participates of both, the direct and the adjoint problem. Numerical results has been obtained and compared with finite-difference evaluations proving the efficiency and the goodness of the method.

References

[1] S.G. Lekhnitskii. *Theory of Elasticity of an Anisotropic Body*. Mir Publishers, Moscow, 1981.

[2] F. París y J. Cañas. *Boundary Element Method. Fundamentals and Applications*. Oxford University Press, 1997.

[3] J.E. Dennis y R.B. Schnabel. *Numerical Methods for Unconstrained Optimization and Nonlinear Equations*. SIAM, 1996.

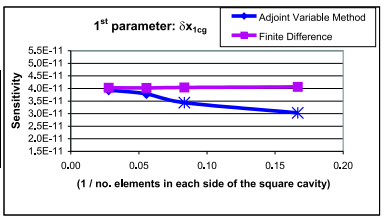
[4] Paullo Sollero, *Fracture Mechanics Analysis of Anisotropic Laminates by the Boundary Element Method*. PhD thesis, Wessex Institute of Technology, 1994.

[5] Guillermo Rus, *Métodos numéricos para la detección no destructiva de defectos* PhD thesis, Grupo de Mecánica de Sólidos y Estructuras, Univ. Granada 2001.

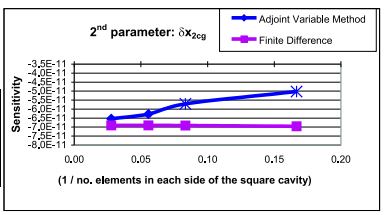
[6] M. Bonnet. *Regularized BIE formulations for first and second-order shape sensitivity of elastic fields*. Computers and Structures, 56, pp.799-811, 1995.

Results for square cavity

number of elements external plate side + internal cavity side		A V M	Fin. Diff.
Derivative parameter δx_{1cg}	40+6	3.227E-11	4.051E-11
	80+12	3.437E-11	4.039E-11
	40+18	3.785E-11	4.026E-11
	80+36	3.933E-11	4.026E-11

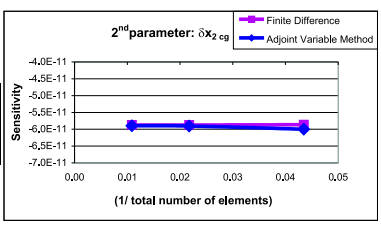


number of elements external plate side + internal cavity side		A V M	Fin. Diff.
Derivative parameter δx_{2cg}	40+6	-5.346E-11	-6.934E-11
	80+12	-5.706E-11	-6.915E-11
	40+18	-6.281E-11	-6.902E-11
	80+36	-6.533E-11	-6.902E-11



Results for circular cavity

number of elements external plate side + internal cavity quarter		A V M	Fin. Diff.
Derivative parameter δx_{2cg}	20+3	-6.00E-11	-5.86E-11
	40+6	-5.905E-11	-5.872E-11
	80+12	-5.895E-11	-5.871E-11



number of elements external plate side + internal cavity quarter		A V M	Fin. Diff.
Derivative parameter $\delta\varepsilon_m$	20+3	2.31E-10	2.54E-10
	40+6	2.439E-10	2.549E-10
	80+12	2.464E-10	2.549E-10

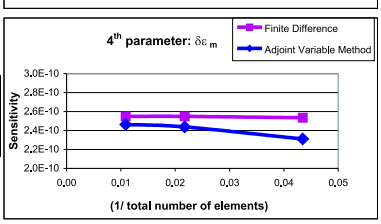


Figure 2: Sensitivity of several shape parameters obtained with AVM compared with Finite Difference results

[7] M. Bonnet. *BIE and material differentiation applied to the formulation of obstacle inverse problems*. Engineering Analysis with BEM,ed. Elsevier, 15, pp.121-136, 1995.

[8] M. Bonnet. *Sensitivity analysis for shape perturbation of cavity or internal cracks using BIE and adjoint variable approach*. International Journal of Solids and Structures,ed. Pergamon, 39, pp.2365-2385, 2002.

[9] R. Gallego y J. Suarez. *Numerical solution of the variation boundary integral equations for inverse problems*.1999

[10] H. Petryc y Z. Mroz. *Time derivatives of integrals functionals defined on varying volume and surface domains*. Arch. Mech., 38, pp.1579-95, 1986.

ISBN 0904 188965

BEM Implementation Of The Biological Growth Method For Structural Shape Optimisation

C. Wessel[1], A. Cisilino[2] and B. Sensale[3]

[1] Facultad de Ingeniería, Universidad Austral, Garay 125, Ciudad Autónoma de Buenos Aires, Argentina, Email: cwessel@fi.mdp.edu.ar
[2] Facultad de Ingeniería, Universidad Nacional de Mar del Plata – CONICET Av. Juan B. Justo 4302 (7600) Mar del Plata, Argentina. Email: cisilino@fi.mdp.edu.ar
[3] Instituto de Estructuras y Transporte, Facultad de Ingeniería, Universidad de la República, J. Herrera y Reissig 665 (11300) Montevideo, Uruguay. Email: sensale@fing.edu.uy

Keywords: Shape Optimization, Biological Growth Method, Boundary Elements Method, Dual Reciprocity Method

Abstract A numerical evolutionary procedure for the structural optimisation for stress reduction of two-dimensional structures is presented in this paper. The proposed methodology is based on the Biological Growth Method (BGM) and implemented using to Boundary Element Method (BEM) formulations: the standard for two-dimensional elastostatics for the stress analysis, and the Dual Reciprocity Method (DRM) for modelling the swelling/shrinking of the optimisation domain. An example is included to illustrate the proposed methodology and to demonstrate its versatility and robustness.

Introduction

The failure of structures under service conditions frequently takes place in areas of locally high stresses. Therefore it is crucial for designers to avoid stress peaks in order to maximize a component service life. Consequently structural shape optimisation for stress reduction has been so far a major issue, examined both with analytical and numerical tools, as reviewed by Belegundu and Chandrupatla [1].

Biological structures, such as bones and trees provide a natural and simple example for shape optimisation, as they change their contour to adapt to external loads while reducing stress peaks. In this line, Mattheck [2] introduced the Biological Growth Method (BGM). Based on his observations in Nature (tree butts, branch joints, deer antlers) he posits that the process of self-optimisation in these structures is carried out through the swelling or shrinking of the soft outermost layer of material that yields the levelling of local stresses.

Since the pioneer work by Mattheck and Burkhardt [3] was published some papers have appeared coupling the BGM with the Finite Element Method (FEM) for structural shape optimisation [4, 5,6,7]. Nevertheless, to the authors'

knowledge there is only one published paper dedicated to coupling BGM and BEM, Cai *et al.* [8]. However in that work, the swelling or shrinking of the material is extended to the complete model and not only to the boundary layer as proposed by Mattheck.
The swelling or shrinking of the soft thin outermost layer can be modelled as a thermoelastic phenomenon, mainly replacing temperature fields by stress distributions as already suggested by Mattheck and Moldenhauer [9]. Thermal effects can be included in the BEM formulation by means of the Dual Reciprocity Method (DRM). Thus the BGM algorithm presented herein combines the standard BEM formulation for elastostatics and DRM to model swelling. The remeshing procedure is carried out by means of exponential splines. An application example is presented to illustrate the proposed methodology and to demonstrate its versatility and robustness.

Biological Growth Method. The Biological Growth Method (BGM) was first introduced by Mattheck [2]. Based on his observations in Nature (tree butts, branch joints, deer antlers, etc.) he posits that biological structures self-optimise their shapes according to natural external loads. He defines *optimum shape* as the one that shows a state of constant stress at part of or the whole of the surface of the component. The process of self-optimisation consists of the swelling or the shrinking of the soft outermost layer of material, following eq (1)

$$\dot{\varepsilon}_v = k\left(\sigma_{vm} - \sigma_{ref}\right) \tag{1}$$

where $\dot{\varepsilon}_v$ is the volumetric swelling strain rate, which is stated to be proportional to a driving function given by the difference from von Mises stress (σ_{vm}) and a reference stress (σ_{ref}), an expected value. This equation holds for each point in the optimisation domain. An elegant method to implement eq (1) is by means of a thermal expansion analogy based on the generalized Hooke's law, where $(\sigma_{vm} - \sigma_{ref})$ is replaced by the change in temperature θ, according to the following eq (2)

$$\alpha\theta = \gamma k\left(\sigma_{vm} - \sigma_{ref}\right) \tag{2}$$

where α is the thermal expansion coefficient, γ is a units conversion factor. Although this is not the only possible approach, it was the strategy followed in this work.
An initial structure shape is proposed, including the thin layer where the optimisation takes place. The Von Mises stresses (σ_{vm}) are calculated both in nodes and internal points using standard BEM. An appropriate choice of a reference stress (σ_{ref}) and the application of the thermal expansion (DRM) given by eq (2) yield the displacements of both nodes and internal points. Once the

ISBN 0904 188965

von Mises stresses at the new co-ordinates have been determined, the optimisation process continues until acceptably low temperature difference values are attained or design limitations restrain further changes in the geometry.

Implementation

The process described in the previous section was implemented using BEM and DRM, so that certain specifications are due:

i. An appropriate BEM mesh is generated for the original and subsequent models using quadratic isoparametric elements. In addition to the boundary nodes, internal collocation points are set. The latter are evenly distributed over the complete model domain and over a thin layer along the optimisation boundary. Typically one to three rows of optimisation internal points are used with this purpose (see Fig 1).

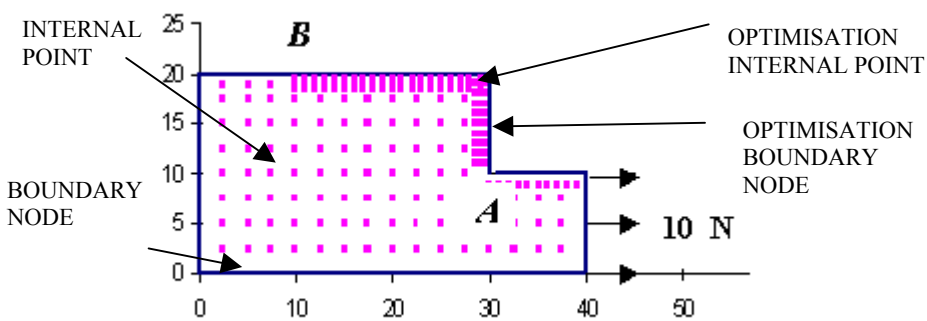


Figure 1. Generation of boundary and internal points on a weld fillet.

ii. Von Mises stresses are computed on the model boundary nodes and on all of the internal points by means of the standard formulation of BEM. Stresses on the boundary points are calculated by numerical differentiation of the displacement field, while for internal points they are directly obtained from their boundary integral representation [10].

iii. A thermal expansion analysis is performed using the DRM formulation with a temperature field θ given by eq (2). In order to limit the swelling to the outermost layer of material, a temperature field different from null is specified only on the optimisation boundary nodes and the optimisation internal points. This computation supplies the displacements along the optimisation boundary. In this work $r^2 log(r)$ was applied and terms up to the second degree in the Pascal (TAPT3 combination, Partridge and Sensale [11]) triangle were chosen as augmentation functions.

iv. The optimisation boundary geometry is updated using a spline interpolation.

The spline interpolation of the new position of the boundary nodes serves two purposes: to smooth the resultant geometry in order to avoid local wrinkles which could act as artificial stress raisers; and to generate a good quality BEM discretization (boundary nodes and internal collocation points) for the new geometry. This process was accomplished using exponential splines.

v. Steps ii to iv are repeated until acceptably low values of σ_{vm} - σ_{ref} are obtained.

Example

The problem presented herein consists in a weld fillet, as already illustrated in fig 1 with the corresponding lengths in cm. A uniform stress σ = 10 MPa is applied in the horizontal direction. The optimisation boundary is indicated with a thick line. Reference stress was chosen as σ_{ref} = 10 MPa. Thirtysix elements and 198 internal points were used in the model discretization. The adopted value for Young modulus was 525MPa. The same problem was solved by Li [6] by means of Sensibility Analysis with FEM.

Twenty-six optimisation loops were necessary in this case. Fig 2 illustrates the evolution of normalized von Mises stresses along the optimisation boundary, where the origin of the abscissas corresponds to position *A*, and 1 to position *B* in Figure 1. Note that with the exception of point *A*, the stress level on most of the optimisation boundary is below the reference value for the original configuration. As the optimisation progresses the thin layer shrinks, what results in a general augmentation in the stress level. Except in the region close to point *B* where stresses can only be null, the final configuration shows a normalized stress distribution approximately equal to reference stress, fact that confirms

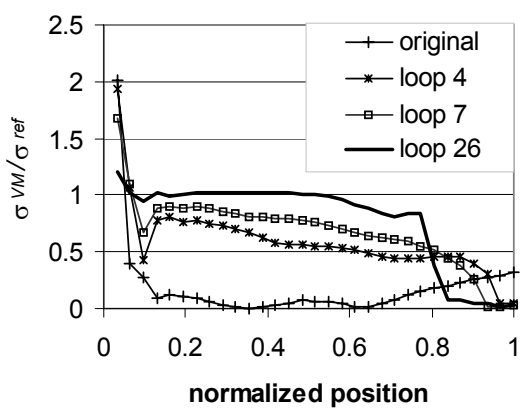


Figure 2. Evolution of normalized von Mises stresses as a function of the normalized position in the optimisation domain.

ISBN 0904 188965

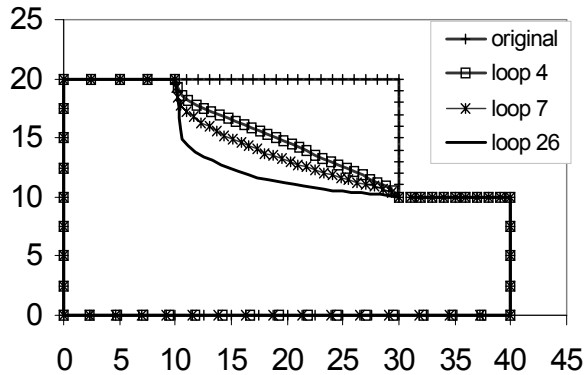


Figure 3. Evolution of fillet geometry as the optimisation

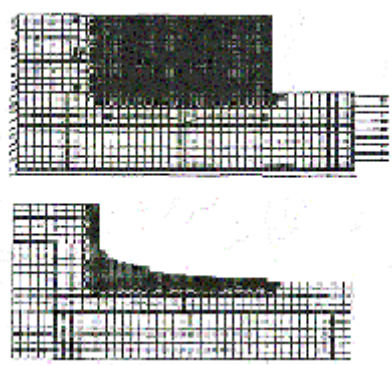


Figure 4. Evolution of fillet geometry as the optimisation progresses, reported by Li *et al*.

that the optimisation procedure has been fulfilled.
The evolution of the model shape is shown in fig 3 for selected optimisation loops. On the other hand, fig 4 illustrates the corresponding results by Li [6]. Note that both methods generate the same final geometry.

Conclusions
A numerical evolutionary procedure for the structural shape optimisation of two-dimensional problems based on the Biological Growth Method was

presented in this work. The proposed methodology proved to be a simple and effective method to obtain homogeneously distributed surface stresses.
Two BEM formulations were combined in this work: the standard elastostatic formulation for the stress analysis, and the DRM for modelling the swelling/shrinking of the optimisation domain. In this way the model domain does not require to be discretized, neither for the stress nor for the swelling/shrinking analyses. This feature made the remeshing a simple task.
The versatility of the proposed methodology was illustrated with one example and excellent results were obtained.

Acknowledgements
This work was founded by grant PICT 12-04586 of Agencia de Promoción Científica de la República Argentina, and the Comisión Sectorial de Investigación Científica de la Universidad de la República de Uruguay.

References
[1] A. D. Belegundu and T. R. Chandrupatla *Optimization concepts and applications in engineering*. New Jersey, USA: Prentice Hall (1999)
[2] C. Mattheck *Design and growth rules for biological structures and their application to engineering, Fat Fract. Engng. Mater Struct*., **13/5**, 535-550 (1990)
[3] C. Mattheck and S. Burkhardt *A new method of structural shape optimisation based on biological growth, Int. J. Fatigue*, **12/3**, 185-190 (1990)
[4] J. L Chen and W. C. Tsai *Shape optimization using simulated biological growth approaches, AIAA journal*, **31/11**, 2143-2147. (1993)
[5] A.E. Tekkaya and A. Güneri *Shape optimization with the biological growth method: a parameter study. Engng Comp*, **13/8**, 4-18 (1998)
[6] Q. Li, P.S. Grant, O.M. Querin and Y.M. Xie *Evolutionary shape optimization for stress minimization, Mech Res Comm*, **26/6** 657-664 (1999)
[7] P. Chaperon *Shape optimization of damage tolerant structures for maximum residual static strength and fatigue Life, BENCHmark*, **01**, 14-16 (2001)
[8] R. Cai, S. Cai, X. Yang and F. Lu *A novel method of structural shape optimization coupling BEM with an optimization method based on biological growth, Struct Opt*, **15**, 296-300 (1998)
[9] C. Mattheck and H. Moldenhauer *An intelligent cad-method based on biological growth*, *Fatigue Fract. Engng. Mater. Struct*, **13/1**, 41-51 (1990)
[10] M.H. Aliabadi and D.P. Rooke *Numerical Fracture Mechanics.* Southampton, UK: Kluwer Academic Publishers (1992)
[11] P. W. Partridge and B. Sensale *Hybrid approximation functions in the dual reciprocity boundary element method*. *Comm Num Meth Engng.* **13,** 83-94

ISBN 0904 188965

Source Identification Using Boundary Element Method with Dual Reciprocity Method

Toshiro Matsumoto[1], Masataka Tanaka[2] and Tomoki Tsukamoto[3]

[1]Department of Mechanical Systems Engineering, Shinshu University
4-17-1, Wakasato, Nagano City, 380-8553, Japan
E-mail: toshiro@gipwc.shinshu-u.ac.jp

[2]Department of Mechanical Systems Engineering, Shinshu University
4-17-1, Wakasato, Nagano City, 380-8553, Japan
E-mail: dtanaka@gipwc.shinshu-u.ac.jp

[3]Graduate School of Shinshu University
4-17-1, Wakasato, Nagano City, 380-8553, Japan

Keywords: Boundary Element Method, Inverse Problem, Source Identification, DRM

Abstract A BEM approach to identify source distributions of Poisson's equation is presented. The domain integral term originated from the source term is converted to boundary integrals by utilizing the dual reciprocity method (DRM). The unknown coefficients of the approximated source term is calculated from the boundary values of the potential and the flux by using the discretized boundary integral equation with DRM. A boundary element code which solves the unknown coefficient from known boundary potentials and fluxes is developed. The effectiveness of the present method is demonstrated through some numerical examples of a two-dimensional source identification problem.

Introduction

Source identification problems are important and found in various mathematical and engineering problems.[1, 2, 3, 4] In this study, a boundary element method based on the dual reciprocity method (DRM) [5] is applied to an identification of source distributions of Poisson's equation. In the usual boundary element formulation, a domain integral term associated with the source term exists and have to be discretized into internal cells in its numerical evaluation. By applying the DRM, the source term in Poisson's equation can be approximated with a linear

combination of radial basis functions and coefficients which, in the direct problem, are determined by using a collocation technique. The domain integral term is then transformed into boundary integrals by using particular solutions of Poisson's equation for the source terms corresponding to the radial basis functions. In the direct problem the coefficient values are determined so that the approximated values of the source fit with those at the boundary nodes and some internal collocation points.

In the present identification problem, the coefficient values become the direct unknowns and to be identified. They are obtained by solving the linear algebraic equation for the unknown coefficients with known boundary values of the potential and the flux. The unknown source values are then calculated by using the obtained coefficients of the DRM approximation. A boundary element code which solves the unknown coefficient from known boundary potentials and fluxes is developed. The effectiveness of the present method is demonstrated through some numerical examples of a two-dimensional source identification problem.

Theory

The boundary integral equation for potential problems with a source term is given by

$$\int_\Gamma q^*(x,y)\,[u(x)-u(y)]\,d\Gamma(x) - \int_\Gamma u^*(x,y)\,q(x)\,d\Gamma(x) = \int_\Omega u^*(x,y)\,b(x)\,d\Omega(x), \quad y \in \Gamma \tag{1}$$

where u is the potential; q is the flux in outward normal direction to the boundary; b is the source; Γ and Ω are the domain and the boundary, respectively. $u^*(x,y)$ is the well-known fundamental solution of the Laplace's equation and $q^*(x,y)$ is its corresponding flux.

To avoid numerical evaluations of the domain integral in eq (1), we convert it to boundary integrals by means of DRM.

DRM uses the following approximation based on the collocation of the source distribution:

$$b(x) = \sum_{l=1}^{N+L} \alpha^l f(x, z^l) \tag{2}$$

where N and L are the numbers of boundary and internal collocation points, respectively, z^l, $(l = 1, 2, \cdots, N+L)$ are the coordinates of the collocation points,

ISBN 0904 188965

and $f(x, z^l)$ are functions of two points x and z^l. Radial basis functions are usually employed as $f(x, z^l)$

Next, let us consider particular solutions $\hat{u}(x, z^l)$, $(l = 1, 2, \cdots, N + L)$ of

$$\nabla^2 \hat{u}(x, z^l) = f(x, z^l), \quad l = 1, 2, \cdots, N + L \tag{3}$$

Then, by utilizing $\hat{u}$ for eqs (2) and (1), the domain integral term for the source term can be converted to boundary integrals, as follows:

$$\int_\Gamma q^*(x, y)\,[u(x) - u(y)]\,d\Gamma(x) - \int_\Gamma u^*(x, y)\,q(x)\,d\Gamma(x)$$
$$= \sum_{l=1}^{N+L} \alpha^l \left(\int_\Gamma q^*(x, y)\,[\hat{u}(x, z^l) - \hat{u}(y, z^l)]\,d\Gamma - \int_\Gamma u^*(x, y)\,\hat{q}(x, z^l)\,d\Gamma \right) \tag{4}$$

By discretizing eq (4) and applying to all the boundary points, we obtain the following system of linear algebraic equations:

$$\mathbf{Hu} - \mathbf{Gq} = \left(\mathbf{H}\hat{\mathbf{U}} - \mathbf{G}\hat{\mathbf{Q}}\right)\boldsymbol{\alpha} \tag{5}$$

In the usual direct problems, the coefficients α^l, $(l = 1, 2, \cdots, N + L)$ are calculated first by using the collocation given by eq (2). The unknowns are the boundary potentials and fluxes which are not given as the boundary condition.

In the present problem, all quantities on the left-hand side of eq (5) are assumed to be known and the coefficient vector $\boldsymbol{\alpha}$ is treated as the unknown. Therefore, the number of unknowns is usually greater than or equal to the number of equations created by applying the discretized boundary integral equation (4) to all the boundary nodes. In the former case, we use singular value decomposition to solve eq (5) for $\boldsymbol{\alpha}$.

Once $\boldsymbol{\alpha}$ is obtained, the source distribution is recovered by using eq (2).

Numerical example

Let us consider a square region as shown in Fig.1. We calculate the source distribution from the boundary values of the potentials and the fluxes shown in Fig.1. The exact distribution of the source in this case is

$$b(x) = b(x_1) = -\frac{1}{24}x^4 + \frac{11}{24}x_1^3 - \frac{13}{12}x_1^2 + \frac{2}{3}x_1 + 1 \tag{6}$$

We discretized the boundary into uniformly arranged 32 quadratic elements. The boundary and internal collocation points for DRM approximation are also uniformly arranged to be $N = 64$ and $L = 225$.

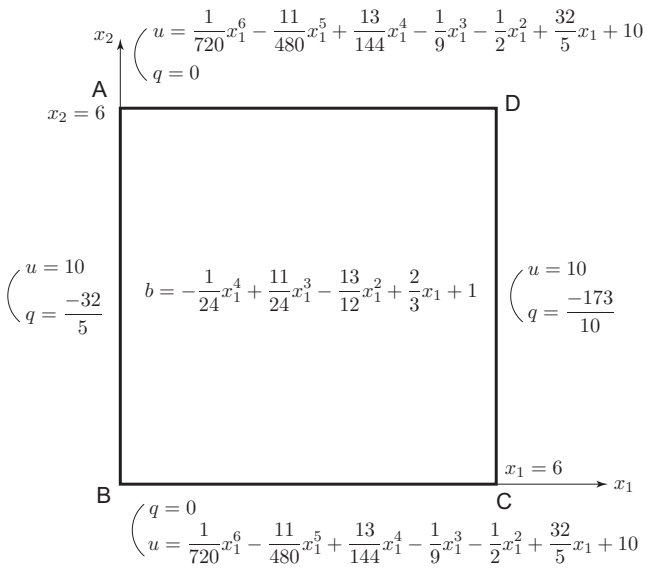


Fig. 1 An example problem in which the boundary values of the potentials and the fluxes are known.

We show in Fig.2 the results for the source distribution along the line $x_2 = 3$. We observe that the source distribution is recovered well from the known data of the boundary potential and flux.

Next we consider a case in which the source values at some internal points shown in Fig.3 are also given in addition to the boundary values of the potential and the flux. We show the results for the identified source distribution along the line $x_2 = 3$ in the square region in Fig.4. The results agree very well with the exact source distribution.

Concluding remarks

In this paper, a boundary element method based on the DRM has been applied to an identification of source distribution of Poisson's equation. The source term has been approximated with a linear combination of radial basis functions and unknown coefficients. The domain integral term is transformed into boundary integrals by using particular solutions of Poisson's equation with the radial basis functions as the source terms.

In the present method, the unknown coefficients are obtained by solving the linear algebraic equation for the unknown coefficients with known boundary values of the potential and flux. The unknown source values are then calculated by

ISBN 0904 188965

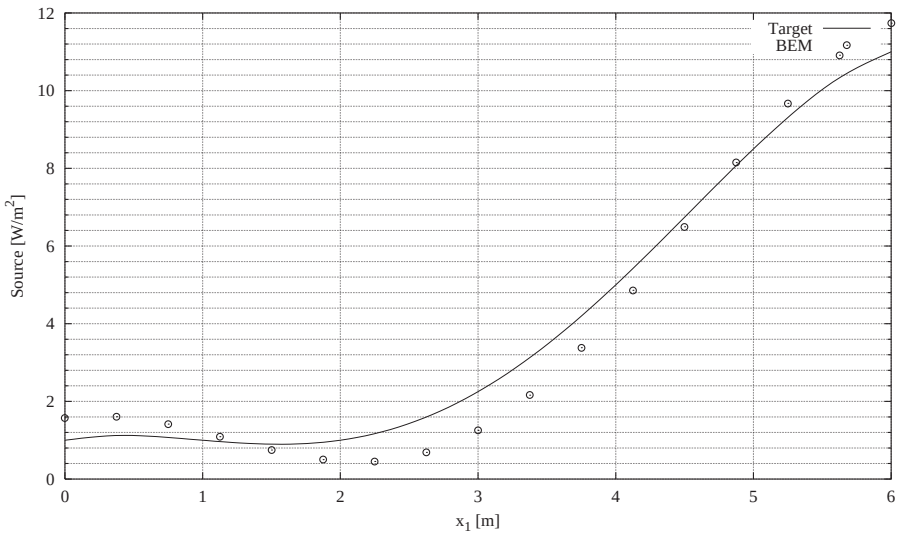


Fig. 2 Results for source distribution along $x_2 = 3$.

using the obtained coefficients of the DRM approximation. The numerical results for example source identification problems have shown good agreements with the exact solution.

References

[1] A. El Badia and T. Ha Duong, Some remarks on the problem of source identification from boundary measurements, Inverse Problems, **14**, (1998), pp. 883–891.

[2] L. Ridolfi and M. Macis, Identification of source terms in nonlinear convection diffusion phenomena by sinc collocation-interpolation methods, *Mathl. Comput. Modelling*, **26**, (1997), pp. 69–79.

[3] P.S. Mahar adn B. Datta, Identification of pollution sources in transient groundwater systems, *Water Resources Management*, **14**, (2000), pp. 209–227.

[4] S. Quinn and J.M. Dulieu-Barton, Identification of the stress of non-adiabatic behaviour for practical thermoelastic stress analysis, *J. Strain Analysis*, **37**, (2002), pp. 59–71.

[5] P. W. Partridge, C. A. Brebbia. L. C. WrobelÅC *The Dual Reciprocity Boundary Element Method*ÅC Computational Mechanics PublicationÅC 1992.

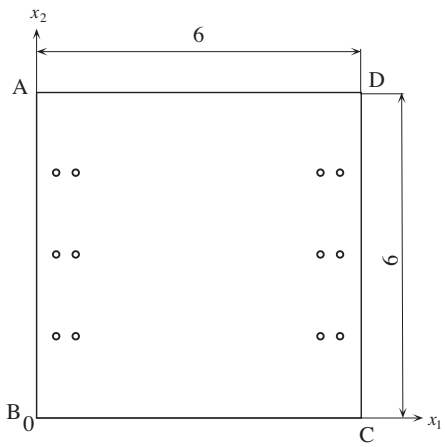


Fig. 3 Internal points at which source values are given.

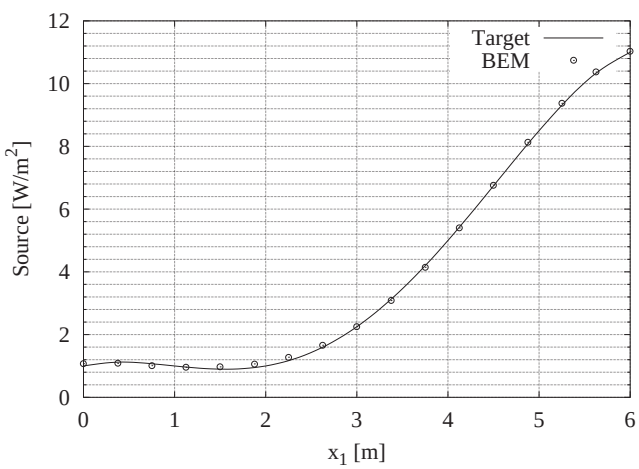


Fig. 4 Results for source distribution along $x_2 = 3$.

ISBN 0904 188965

A New Error Estimator Based on Tangential Residuals in BEM

Alejandro E. Martínez-Castro[1] and Rafael Gallego[2]

[1] Dept. Structural Mechanics, University of Granada, Politécnico de Fuentenueva, 18071 Granada, Spain, amcastro@ugr.es

[2] Dept. Structural Mechanics, University of Granada, Politécnico de Fuentenueva, 18071 Granada, Spain, gallego@ugr.es

Keywords: Error Estimation, Residual, Sensitivity Analysis, Mesh Adaptation, Mesh Refinement, Hypersingular, High-Order singular, Boundary Element Techniques, BEM

Abstract A new error estimator based on residuals is presented. Residual of the Tangential Boundary Integral Equation (TBIE) leads to an error estimator for the tangential derivative of the considered variable (flux or potential). The Laplace and Helmholtz equations are considered, in a bi-dimensional domain. The analysis have been carried out for both standard and hypersingular formulations. Theoretical aspects show that the tangential residual can be related with the error in the tangential derivative. A similar relation is done with nodal sensitivity analysis. Numerical tests shows that this error estimator based on tangential residual is accurate, and the computational cost is lower than nodal sensitivity analysis.

Introduction

Recently, Kita and Kamiya [1] have presented a general review on error estimation and mesh adaptation. An adaptive scheme consists of three processes: the error estimation, the adaptive tactics, and the mesh refinement. But the most important of them is obviously the error estimator. Error estimation schemes are classified in residual type, interpolation error type, boundary integral error type, the node-sensitivity type and the solution difference type. In this work a new error estimator, of "residual" type is presented.

The residual of a Boundary Integral Equation can be defined as the different between the value of the approximated solution and the value obtained with the BIE. In the collocation method of the BEM, this residual is zero at a set of *collocation points.* If an error exists in the value of a variable, it is expected that the residual will not be zero. Based on this idea, Cerrolaza and Alarcón [2] defined a residual error estimator. Paulino *et al.* [3, 4] defined an error indicator based on hypersingular residuals.

Both error estimators are "pointwise", in the sense that they predict errors obtained in the variable calculated. Following Guiggiani [5] two displacement fields (potential) can be considered truly different if their tangential derivatives are different. Two types of error must be considered here: error at nodal values and error in the shape function. Only the second error is important, and must be detected. This error type leads to the requirements of new shape functions (-*h*, -*p*, -*hp* types).

Sensitivity analysis (Paulino [6], Guiggiani [5] and Bonnet [7]) is a technique that leads to a new error estimator. The rate of change of the nodal values respect to the position of a set of points, with the same geometrical mesh, leads to the definition of Nodal Sensitivities. The calculus of these N.S. requires the evaluation of tangential residuals and the solution of a linear system. Paulino related these nodal values with tangential errors (based on interpolation ideas). Guiggiani *et al.* have developed the Direct Differentiation Approach for the BEM. Keeping on this idea, Gallego *et al.* [8] presented sensitivity analysis for the hypersingular equation. Recently, Miranda-Valenzuela and Muci-Küchler [9] have used tangential derivative to obtain an error estimator based on two solutions difference.

In this work, a new error estimator based on tangential residual is presented. It is shown a theoretical relation with error obtained in the tangential derivative of the variable calculated, for the BEM mesh. A similar relation is shown for nodal sensitivity. Numerical tests confirms that both indicators are error estimators for tangential derivative error function, with better results and with low computational effort for tangential residual.

Tangential residual and tangential error

Let be Ω a 2D domain. Let be Γ the boundary of Ω. Let be $\boldsymbol{x}$ a boundary point, and $\boldsymbol{y}$ the collocation point; let be u the potential, and q the normal derivative (flux). Considering the properly expressions for kernel functions u^*, q^*, d^*, s^*,, the following BIEs can be stablished for a general Laplace or Helmholtz problem (smooth boundary point):

$$\frac{u(\boldsymbol{y})}{2} + \int_\Gamma q^*(\boldsymbol{y},\boldsymbol{x})\, u(\boldsymbol{x})\, d\Gamma(\boldsymbol{x}) = \int_\Gamma u^*(\boldsymbol{y},\boldsymbol{x})\, q(\boldsymbol{x})\, d\Gamma(\boldsymbol{x}) \tag{1}$$

$$\frac{q(\boldsymbol{y})}{2} + \int_\Gamma d^*(\boldsymbol{y},\boldsymbol{x})\, q(\boldsymbol{x})\, d\Gamma(\boldsymbol{x}) = \int_\Gamma s^*(\boldsymbol{y},\boldsymbol{x})\, u(\boldsymbol{x})\, d\Gamma(\boldsymbol{x}) \tag{2}$$

In order to obtain a stable second kind integral equation, eq (1) is used at collocation points whose unknown data is the potential; for points with unknown

ISBN 0904 188965

values in the flux, eq (2) must be used. Considering these equations, tangential BIEs can be obtained:

$$\frac{u_{\boldsymbol{T}}(\boldsymbol{y})}{2} + \int_{\Gamma} q^*_{\boldsymbol{T}}(\boldsymbol{y},\boldsymbol{x})\, u(\boldsymbol{x})\, d\Gamma(\boldsymbol{x}) = \fint_{\Gamma} d^*_{\boldsymbol{T}}(\boldsymbol{y},\boldsymbol{x})\, q(\boldsymbol{x})\, d\Gamma(\boldsymbol{x}) \tag{3}$$

$$\frac{q_{\boldsymbol{T}}(\boldsymbol{y})}{2} + \int_{\Gamma} d^*_{\boldsymbol{T}}(\boldsymbol{y},\boldsymbol{x})\, q(\boldsymbol{x})\, d\Gamma(\boldsymbol{x}) = \fint_{\Gamma} s^*_{\boldsymbol{T}}(\boldsymbol{y},\boldsymbol{x})\, u(\boldsymbol{x})\, d\Gamma(\boldsymbol{x}) \tag{4}$$

Both equations, eq (3) and eq (4) are equally valid, in order to obtain numerical solutions. Let us consider eq (3) and let us define the residual. Fields u and q can be decomposed in two adding terms: approximed solutions and error functions. This decomposition is equally valid for the tangential formulations. Thus:

$$u = \hat{u} + e_u \; ; \; q = \hat{q} + e_q \tag{5}$$

$$u_{\boldsymbol{T}} = \hat{u_{\boldsymbol{T}}} + e_{u_{\boldsymbol{T}}} \; ; \; q_{\boldsymbol{T}} = \hat{q_{\boldsymbol{T}}} + e_{q_{\boldsymbol{T}}} \tag{6}$$

Considering these relations, the tangential BIE for potential can be rewritten and grouped as:

$$\begin{aligned} &\frac{\hat{u_{\boldsymbol{T}}}(\boldsymbol{y})}{2} + \int_{\Gamma} q^*_{\boldsymbol{T}}(\boldsymbol{y},\boldsymbol{x})\, \hat{u}(\boldsymbol{x})\, d\Gamma(\boldsymbol{x}) - \fint_{\Gamma} d^*_{\boldsymbol{T}}(\boldsymbol{y},\boldsymbol{x})\, \hat{q}(\boldsymbol{x})\, d\Gamma(\boldsymbol{x}) = \\ &-\left(\frac{e_{u_{\boldsymbol{T}}}(\boldsymbol{y})}{2} + \int_{\Gamma} q^*_{\boldsymbol{T}}(\boldsymbol{y},\boldsymbol{x})\, e_u(\boldsymbol{x})\, d\Gamma(\boldsymbol{x}) - \fint_{\Gamma} d^*_{\boldsymbol{T}}(\boldsymbol{y},\boldsymbol{x})\, e_q(\boldsymbol{x})\, d\Gamma(\boldsymbol{x}) \right) \end{aligned} \tag{7}$$

Tangential residual, $\varepsilon_{u_{\boldsymbol{T}}}$, can be defined as:

$$-\frac{\varepsilon_{u_{\boldsymbol{T}}}}{2} = \frac{\hat{u_{\boldsymbol{T}}}(\boldsymbol{y})}{2} + \int_{\Gamma} q^*_{\boldsymbol{T}}(\boldsymbol{y},\boldsymbol{x})\, \hat{u}(\boldsymbol{x})\, d\Gamma(\boldsymbol{x}) - \fint_{\Gamma} d^*_{\boldsymbol{T}}(\boldsymbol{y},\boldsymbol{x})\, \hat{q}(\boldsymbol{x})\, d\Gamma(\boldsymbol{x}) \tag{8}$$

Considering eq (7), and neglecting integral terms, it can be written:

$$\varepsilon_{u_{\boldsymbol{T}}} \simeq e_{u_{\boldsymbol{T}}} \tag{9}$$

Considering hypersingular formulation, and tangential hypersingular BIE, eq (4), a similar result can relate tangential residual with error in the tangential direction.

$$\varepsilon_{q_{\boldsymbol{T}}} \simeq e_{q_{\boldsymbol{T}}} \tag{10}$$

Considering the decomposition in an approximated solution plus an error function, it is possible to rewritte integral terms in eq (8) (and hypersingular tangential BIE) and find the expressions of the terms neglected in eq () and eq (10).

Thus:

$$\frac{\varepsilon_{u_{\boldsymbol{T}}}}{2} = \left(\frac{e_{u_{\boldsymbol{T}}}}{2} + \left[\frac{u_{\boldsymbol{T}}}{2}\Big|_{BIE} - \frac{u_{\boldsymbol{T}}}{2}\right]\right) \tag{11}$$

$$\frac{\varepsilon_{q_{\boldsymbol{T}}}}{2} = \left(\frac{e_{q_{\boldsymbol{T}}}}{2} + \left[\frac{q_{\boldsymbol{T}}}{2}\Big|_{HBIE} - \frac{q_{\boldsymbol{T}}}{2}\right]\right) \tag{12}$$

The difference between exact tangential derivative and tangential derivative obtained from the BIE is close to zero. Numerical tests confirms this idea. So integral terms can be neglected, and tangential residual is an error estimator for tangential derivative error function.

Relation between tangential residual, nodal sensitivity, and tangential error

Calculus of nodal sensitivities requires two previous steps: calculus of the tangential residual and solution of a linear system. Let be $\tilde{u}$ and $\tilde{q}$ nodal sensitivityes. The following equations are stablished at *perturbation points*:

$$-\frac{\varepsilon_{u_{\boldsymbol{T}}}}{2} = \frac{\tilde{u}(\boldsymbol{y})}{2} + \int_\Gamma q^*(\boldsymbol{y},\boldsymbol{x})\,\tilde{u}(\boldsymbol{x})\,d\Gamma(\boldsymbol{x}) - \int_\Gamma u^*(\boldsymbol{y},\boldsymbol{x})\,\tilde{q}(\boldsymbol{x})\,d\Gamma(\boldsymbol{x}) \tag{13}$$

$$-\frac{\varepsilon_{q_{\boldsymbol{T}}}}{2} = \frac{\tilde{q}(\boldsymbol{y})}{2} + \int_\Gamma d^*(\boldsymbol{y},\boldsymbol{x})\,\tilde{q}(\boldsymbol{x})\,d\Gamma(\boldsymbol{x}) - \fint_\Gamma s^*(\boldsymbol{y},\boldsymbol{x})\,\tilde{u}(\boldsymbol{x})\,d\Gamma(\boldsymbol{x}) \tag{14}$$

Considering eq () and eq (), it can be written:

$$-\frac{e_{u_{\boldsymbol{T}}}}{2} \simeq \frac{\tilde{u}(\boldsymbol{y})}{2} + \int_\Gamma q^*(\boldsymbol{y},\boldsymbol{x})\,\tilde{u}(\boldsymbol{x})\,d\Gamma(\boldsymbol{x}) - \int_\Gamma u^*(\boldsymbol{y},\boldsymbol{x})\,\tilde{q}(\boldsymbol{x})\,d\Gamma(\boldsymbol{x}) \tag{15}$$

$$-\frac{e_{q_{\boldsymbol{T}}}}{2} \simeq \frac{\tilde{q}(\boldsymbol{y})}{2} + \int_\Gamma d^*(\boldsymbol{y},\boldsymbol{x})\,\tilde{q}(\boldsymbol{x})\,d\Gamma(\boldsymbol{x}) - \fint_\Gamma s^*(\boldsymbol{y},\boldsymbol{x})\,\tilde{u}(\boldsymbol{x})\,d\Gamma(\boldsymbol{x}) \tag{16}$$

And, neglecting integral terms, tangential residual can be related with nodal sensitivity and with error tangential derivative:

$$\varepsilon_{u_{\boldsymbol{T}}} \simeq e_{u_{\boldsymbol{T}}} \simeq \tilde{u}\,;\; \varepsilon_{q_{\boldsymbol{T}}} \simeq e_{q_{\boldsymbol{T}}} \simeq \tilde{q} \tag{17}$$

Numerical results

A set of numerical tests have been done in order to validate both error estimators. Two domains, a rectangular plate and a disc sector, have been considered. For each one, Error Tangential Derivative (ETD), exact, has been compared with Tangential Residual (TD) and Nodal Sensitivity (NS). Residual has been computed at mid-nodes of quadratic boundary elements.

ISBN 0904 188965

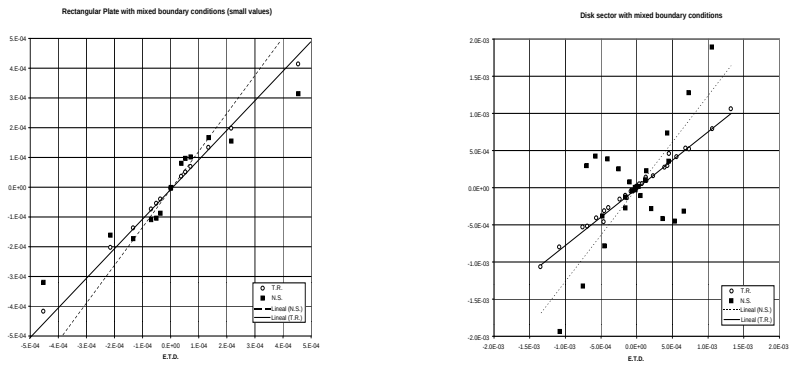


Figure 1: Error estimators. Laplace equation. Left: Rectangular plate with mixed B.C. Right: Disc Sector with mixed B.C.

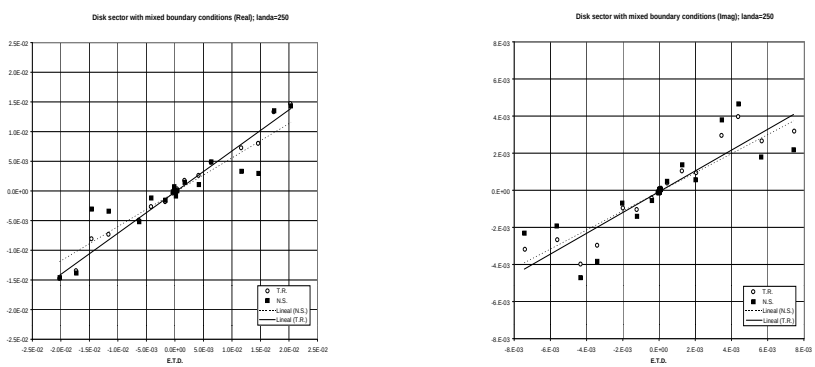


Figure 2: Error estimators. Helmholtz equation. Left: Disc Sector with mixed B.C. (RE). Right: Disc Sector with mixed B.C. (Im).

Summary and conclusion

A new error estimator based on tangential residual has been proposed, for the boundary element method. Error estimator is the tangential residual. It predicts accurately the tangential error function. The computational cost is low. Nodal Sensitivity has been related with the same type of error, and it is concluded that it is an error estimator with high computational cost (compared with TR). The accuracy of the predicted values with TR is better than with NS.

References

[1] E. Kita and N. Kamiya. *Error estimation and adaptive mesh refinement in boundary element method, an overview. Engineering Analysis with Boundary Elements*, 25:479–495, 2001.

[2] M. Cerrolaza and E. Alarcón. *p-adaptive boundary elements for three dimensional potential problems. Communications in Applied Numerical Methods*, 3:335–344, 1987.

[3] Glaucio H. Paulino, L.J. Gray, and V. Zarikian. *Hypersingular residuals-a new approach for error estimation in the boundary element method. Int. J. Numerical Methods in Engineering*, 39:2005–2029, 1996.

[4] G. Menon, G.H. Paulino, and S. Mukerjee. *Analysis of hypersingular residual error estimates in boundary element methods for potential problems. Computer Methods in Applied Mechanics and Engineering*, 1868:449–473, 1999.

[5] M. Guiggiani. *Sensitivity analysis for boundary element. Error estimation and mesh refinement. Int. J. Numerical Methods in Engineering*, 39:2907–2920, 1996.

[6] G. H. Paulino. *Novel formulations of the boundary element method for fracture mechanics and error estimation.* PhD thesis, Cornell University, August 1995.

[7] M. Bonnet and M. Guiggiani. *Tangential derivative of singular boundary integrals with respecto to the position of collocation points. Int. J. Numerical Methods in Engineering*, 41:1255–1275, 1998.

[8] R. Gallego, A. Martínez-Castro, and J.Suárez. *Sensitivity analysis of approximate solutions for the hypersingular boundary element method.* In *Proceedings of the Boundary Elements Techinques Conference*, pages 311–319, London, 1999. M.H. Aliabadi.

[9] J. C. Miranda-Valenzuela and K.H. Muci-Kückler. *Adaptive Meshing with Boundary Elements.* Wiley, 2001.

ISBN 0904 188965